

HOMOGENEOUS MODELS OF C_3 MONGE GEOMETRIES

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1. PRELIMINARIES

1.1. Introduction. *Distributions of Monge type* are a class of strongly regular bracket-generating distributions introduced by I. Anderson, Zh. Nie and P. Nurowski in [1]. Their symbol algebras prolong to semisimple graded Lie algebras, thus allowing one to associate a *parabolic geometry* to any given *Monge distribution*. This article is devoted to the classification problem for homogeneous models of Monge distributions of type C_3 in dimension eight. These are rank 3 distributions in dimension 8, whose symbol algebra prolongs to the semisimple Lie algebra $\mathfrak{sp}(6, \mathbb{R})$ with a suitable grading. Applying the techniques of parabolic geometry, one associates to every such distribution \mathcal{D} a pair of invariants (components of the harmonic curvature): a ‘scalar’, and a ‘quintic’. The scalar invariant is a section of a natural line bundle; the quintic one is a section of $S^5 \mathcal{D}^*$ twisted by a natural line bundle. We restrict the classification problem to at least 2-transitive homogeneous C_3 Monge distributions whose scalar invariant vanishes. The general classification algorithm, as well as most of its application to the particular problem, are joint work with Ian Anderson (USU Logan); its formalisation in this paper is a sole responsibility of the present author.

The classification algorithm we use had been developed by Ian Anderson and the author in the context of classifying homogeneous models of strongly regular distributions whose symbol prolongs to a semi-simple Lie algebra \mathfrak{g} . It relies on a reformulation of the classification problem in terms of filtered deformations of certain graded subalgebras of \mathfrak{g} , at the same time using parabolic geometry to compute the invariants (harmonic curvature) and symmetries. Necessary techniques of deformation theory and parabolic geometry have been implemented in the `DifferentialGeometry` package for MAPLE. An in-depth technical presentation will appear as a separate work (in preparation). Here we do not touch upon these technical details.

On the other hand, a significant amount of space is devoted to showing that the conceptual foundations of our classification method are sound. Since we pass through several reformulations of a classification problem, it becomes crucial to carefully keep track not only of the objects being classified, but also of the notion of equivalence. While there are other ways to do it, we subscribe to the view that a categorical formalisation is the most convenient and reliable bookkeeping device. Hence we use the categorical language, perhaps to a somewhat larger extent than typical for this area. We hope that, as intended, the reader will find it an aid rather than an obstacle.

The present paper is to be regarded as forming a pair with a parallel development due to Ian Anderson and Paweł Nurowski [2]. The latter work approaches the same classification problem, although covering a slightly different area: it does not restrict to at least 2-transitive models, but, on the other hand, considers only the case where the quintic invariant is a fifth power (i.e., type N below). Importantly, Anderson and Nurowski use a very different framework, namely the method of *Cartan reduction*. There, a generic Monge geometry is viewed as an abstract exterior differential system on a principal bundle; one then exploits the action of the structure group to normalise the structure functions (curvatures, torsions), gradually reducing the bundle. As each normalisation step may introduce a number of branches (corresponding to the orbits of the structure group in some space of structure coefficients), one eventually obtains a tree whose leaves carry the various families of homogeneous models; tracing the path from a leaf to the root, one may label these families by the normalisation conditions. Thus, in the course of classifying the homogeneous models, Anderson and Nurowski produce also the invariants for different classes of models, as well as explicit Cartan connections with strong normalisations. Furthermore, they provide realisations of Monge geometries in terms of ordinary differential equations, and identify the structure invariants of a Cartan geometry with the differential invariants of an ODE – an aspect we do not touch upon in the present paper. We shall comment further on the relation between the two papers in the final section.

1.2. Acknowledgements. I am immensely grateful to Professor Ian Anderson for a rewarding collaboration and his hospitality during my stay at USU Logan in Spring 2015. The present article is to a very large degree based on our joint work. I am likewise indebted to Professor Paweł Nurowski for inviting me to join the project, providing an excellent working environment, and generously sharing the details of his own approach to the problem. I acknowledge the support of the Polish National Science Centre (NCN) grant DEC-2013/09/B/ST1/01799.

1.3. Lie-theoretic setup. Throughout the paper, we let $\mathfrak{g} \simeq \mathfrak{sp}(6, \mathbb{R})$ be the split real form of the semisimple Lie algebra of type C_3 . Fixing a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we have the root system $\Phi \subset \mathfrak{h}^*$, a subset of positive roots $\Phi^+ \subset \Phi$ and simple roots $\Delta \subset \Phi^+$. We use Bourbaki labelling $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ where α_3 is long; let $\varpi_1, \varpi_2, \varpi_3 \in \mathfrak{h}^*$ be the corresponding fundamental weights so that $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$. The height function

$$\text{ht} : \Phi \rightarrow \mathbb{Z}, \quad \text{ht}(\alpha) = \langle \varpi_2 + \varpi_3, \alpha^\vee \rangle.$$

induces a grading \mathfrak{g}_\bullet such that

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\text{ht}(\alpha)=0} \mathfrak{g}_\alpha, \quad \mathfrak{g}_i = \bigoplus_{\text{ht}(\alpha)=i} \mathfrak{g}_\alpha, \quad i \neq 0.$$

This turns \mathfrak{g} into a graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3.$$

We also use the filtration \mathfrak{g}^\bullet where $\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$, with a natural identification $\text{gr } \mathfrak{g}^\bullet = \mathfrak{g}_\bullet$. In particular, $\mathfrak{p} = \mathfrak{g}^0$ is a parabolic subalgebra, $\mathfrak{p}_+ = \mathfrak{g}^1$ its nilpotent radical, and \mathfrak{g}_0 the Levi factor.

Since \mathfrak{g} is semi-simple, the grading \mathfrak{g}_\bullet is induced by a unique $E \in \mathfrak{h}$, called the *grading element*, such that \mathfrak{g}_i is the eigenspace of ad_E with eigenvalue i . This allows us to induce a compatible grading on any finite-dimensional $U(\mathfrak{h})$ -module V , defining V_i to be the eigenspace of $E \cdot$ with eigenvalue i . Again, we $V^i = \bigoplus_{j \leq i} V_j$ so that V^\bullet is a filtration such that $\text{gr } V^\bullet = V_\bullet$ naturally (as $U(\mathfrak{h})$ -modules). If V is a representation of \mathfrak{g}_0 , the grading V_\bullet and the filtration V^\bullet are \mathfrak{g}_0 -equivariant. If V is a representation of \mathfrak{p} , the filtration V^\bullet is \mathfrak{p} -equivariant.

1.4. Monge distributions. We let $\mathfrak{g}_- = \bigoplus_{i < 0} \mathfrak{g}_i$, a graded nilpotent Lie algebra. As is shown in [ANN], \mathfrak{g} is precisely the Tanaka prolongation of \mathfrak{g}_- . In particular, \mathfrak{g}_0 is the Lie algebra of derivations of \mathfrak{g}_- ; we define $G_0 = \text{Aut } \mathfrak{g}_-$. Note that G_0 acts naturally on \mathfrak{g} , and the action preserves the grading.

The ranks of the relevant graded subspaces are as follows:

$$\dim \mathfrak{g}_{-3} = 3, \quad \dim \mathfrak{g}_{-2} = 2, \quad \dim \mathfrak{g}_{-1} = 3, \quad \dim \mathfrak{g}_0 = 5.$$

Following op. cit., we have G_0 -equivariant identifications

$$\begin{aligned} \mathfrak{g}_{-1} &= \mathfrak{r} \oplus \mathfrak{a} & \mathfrak{g}_0 &= \text{End } \mathfrak{r} \oplus \text{End } \mathfrak{a} \\ \mathfrak{g}_{-2} &= \mathfrak{r} \otimes \mathfrak{a} & G_0 &= \text{GL}(\mathfrak{r}) \times \text{GL}(\mathfrak{a}) \\ \mathfrak{g}_{-3} &= \mathfrak{r} \otimes S^2 \mathfrak{a} \end{aligned}$$

where \mathfrak{r} is the root subspace of $-\alpha_3$, \mathfrak{a} is a rank 2 abelian subalgebra in \mathfrak{g}_- , and the Lie bracket is given by the natural G_0 -equivariant projections.

Definition 1. A C_3 Monge distribution is a strongly regular rank 3 distribution on an 8-dimensional manifold, whose symbol is isomorphic to \mathfrak{g}_- .

1.5. Associated parabolic geometries. Let P_+ be the connected, simply-connected unipotent Lie group with Lie algebra \mathfrak{p}_+ , and set $P = G_0 \ltimes P_+$ with respect to the natural G_0 -action on $\mathfrak{p}_+ \simeq \mathfrak{g}_-^*$.

Proposition 1. *There is an equivalence of categories between:*

- (1) *the category of C_3 Monge distributions and local equivalences,*
- (2) *the category of regular, normal parabolic geometries of type (\mathfrak{g}, P) , and local equivalences.*

Proof. This follows from Theorem 3.1.14 of [6], where we refer to [1] for the fact that $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = 0$. \square

Here a *local equivalence* between a pair of distributions $\mathcal{D} \subset TM$ and $\mathcal{E} \subset TN$ is a local diffeomorphism $f : M \rightarrow N$ such that $f^{-1}\mathcal{E} = \mathcal{D}$. A *parabolic geometry* of type (\mathfrak{g}, P) is a P -principal bundle $\pi : \mathcal{G} \rightarrow M$ together with a \mathfrak{g} -valued Cartan connection ω inducing an isomorphism of π^*TM with the trivial bundle with fibre

$\mathfrak{g}/\mathfrak{p}$. It is *regular* if its curvature function $\kappa : \mathcal{G} \rightarrow \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$, factors through the degree 1 subspace for the \mathfrak{p} -equivariant filtration induced by the grading element. It is *normal* if the curvature function factors through the space of cycles $Z_2(\mathfrak{p}_+, \mathfrak{g})$ for the natural codifferential on the chain complex $C_\bullet(\mathfrak{p}_+, \mathfrak{g})$ computing Lie algebra homology of \mathfrak{p}_+ with values in \mathfrak{g} (using the adjoint representation). Finally, a *local equivalence* between a pair of parabolic geometries $(\mathcal{G}/M, \omega)$ and $(\mathcal{G}'/N, \omega')$ is a local diffeomorphism $f : M \rightarrow N$ together with a P -principal bundle isomorphism $\tilde{f} : \mathcal{G} \rightarrow f^* \mathcal{G}'$ over M such that the induced map $\mathcal{G} \rightarrow \mathcal{G}'$ pulls back ω' to ω .

We refer to [6] for the necessary background on Cartan and in particular parabolic geometries. Let us recall that viewing the curvature function of a regular, normal parabolic geometry as a function

$$\kappa : \mathcal{G} \rightarrow Z_2(\mathfrak{p}_+, \mathfrak{g})^1$$

we may consider its composite with projection to homology: that is the *harmonic curvature*

$$\kappa_H : \mathcal{G} \rightarrow H_2(\mathfrak{p}_+, \mathfrak{g})^1$$

where the upper index \cdot^1 refers to the \mathfrak{p} -equivariant filtration induced by the grading element. The homology space is a completely reducible representation of \mathfrak{p} , so that \mathfrak{p}_+ acts trivially, and the action factors through \mathfrak{g}_0 . The celebrated theorem of Kostant allows one to find the highest weights of the simple $U(\mathfrak{g}_0)$ -submodules. One then finds that

$$H_2(\mathfrak{p}_+, \mathfrak{g})^1 \simeq \mathbb{R}[-10, 4] \oplus S^5 \mathfrak{a}^* \otimes \mathbb{R}[-3, 1]$$

where $\mathbb{R}[p, q] \simeq \mathfrak{x}^{-q} \otimes (\det \mathfrak{a})^{\frac{q-p}{2}}$ is a one-dimensional representation of G_0 (provided $p+q$ is even). In particular restricting to the semisimple part of G_0 , isomorphic to $\mathrm{SL}(2, \mathbb{R})$, the two summands correspond to scalars and quintic binomials, respectively. Accordingly, the harmonic curvature κ_H decomposes into what we referred to as the scalar and quintic component.

Observe that the harmonic curvature as an $H_2(\mathfrak{p}_+, \mathfrak{g})^1$ -valued function on \mathcal{G} factors through the G_0 -principal bundle $\mathcal{G}_0 = \mathcal{G}/P_+$. Being P -equivariant, it may then be viewed as a G_0 -equivariant map $\mathcal{G}_0 \rightarrow H_2(\mathfrak{p}_+, \mathfrak{g})^1$, and thus a section of the associated vector bundle over M . Since \mathcal{G}_0 may be identified with an adapted frame bundle for \mathcal{D} , it follows that the associated bundle $\mathcal{G}_0 \times^{G_0} H_2(\mathfrak{p}_+, \mathfrak{g})^1$ is a certain tensor bundle; thus, κ_H may be viewed as a tensorial invariant defined on M .

1.6. Classification problem. We may now state the classification problem in a precise way: *classify local equivalence classes of Monge C_3 distributions such that (1) the distribution admits a transitive symmetry group with at least two-dimensional isotropy, (2) the scalar component of the harmonic curvature of the corresponding parabolic geometry of type (\mathfrak{g}, P) vanishes.*

We follow the idea that an equivalence problem (including auto-equivalences, i.e. symmetries) is naturally organised into a groupoid, i.e. a small category whose arrows are isomorphisms (without the smallness assumption, we have a ‘large groupoid’). The classification problem amounts to describing the set of isomorphism classes of objects, and the latter is invariant under equivalence of categories. In particular, even if the category we begin with is large, we may pass to an equivalent small groupoid.

In the case at hand, we shall set up such framework for a slightly more general problem, namely that of classifying *all* locally homogeneous Monge C_3 distributions up to local equivalence. We begin with the following category of *pointed* locally homogeneous models.

Definition 2. Model is the category whose objects are triples (M, \mathcal{D}, m) where $\mathcal{D} \subset TM$ is a C_3 Monge distribution with an *infinitesimally transitive* symmetry algebra, and $m \in M$ is a point. Morphisms $(M, \mathcal{D}, m) \rightarrow (N, \mathcal{E}, n)$ are local diffeomorphisms $f : M \rightarrow N$ such that $f^{-1}\mathcal{E} = \mathcal{D}$ and $f(m) = n$.

Thus defined, **Model** is not a (large) groupoid, but we can turn it into one by formally inverting its morphisms. Recall that given a category \mathbf{C} , its *localisation* at the collection $\text{Mor } \mathbf{C}$ of all morphisms is the large groupoid $\text{Frac } \mathbf{C} = \mathbf{C}[(\text{Mor } \mathbf{C})^{-1}]$, together with a functor $\mathbf{C} \rightarrow \text{Frac } \mathbf{C}$, satisfying a universal property with respect to functors from \mathbf{C} to large groupoids. If \mathbf{C} has pullbacks, one may represent the arrows $a \rightarrow c$ of $\text{Frac } \mathbf{C}$ by diagrams $a \leftarrow b \rightarrow c$ in \mathbf{C} (two such diagrams define the same arrow if they both ‘factor’ a third such diagram). Note that **Model** does have pullbacks, whence $\text{Frac } \mathbf{Model}$ is simply the category of triples (M, \mathcal{D}, m) where a morphism to (N, \mathcal{E}, n) is a diagram $M \leftarrow \tilde{M} \rightarrow N$ of pointed local equivalences.

It is intuitively clear that, having inverted pointed local diffeomorphisms, we might as well work with germs of distribution at a point o of the fixed manifold \mathbb{R}^8 . These are defined in the usual way as equivalence classes \mathcal{D} of pairs (U, \mathcal{D}_U) where U is an open neighbourhood of o and $\mathcal{D}_U \subset TU$ a distribution. The *symmetry algebra* $\text{sym } \mathcal{D}$ is then understood to be the local symmetry algebra at o of any representative, viewed as a Lie algebra of germs of vector fields. An *equivalence* of germs \mathcal{D} and \mathcal{E} is a germ at o of a diffeomorphism $\mathbb{R}^8 \rightarrow \mathbb{R}^8$ sending o to itself and \mathcal{D} to \mathcal{E} .

Definition 3. Germ is the groupoid whose objects are germs at o of Monge C_3 distributions \mathcal{D} on \mathbb{R}^8 such that the evaluation map $\text{ev}_o : \text{sym } \mathcal{D} \rightarrow T_o\mathbb{R}^8$ is surjective, and whose morphisms are equivalences as defined above.

Proposition 2. *There is an equivalence of categories between **Germ** and $\text{Frac } \mathbf{Model}$.*

The consequence is that the local equivalence problem for locally homogeneous Monge C_3 distributions is encoded by the small groupoid **Germ**. In particular, the original classification problem stated in the beginning of this subsection reduces to the problem of describing isomorphism classes of objects of a suitable sub-groupoid of **Germ** (consisting of germs \mathcal{D} with $\ker \text{ev}_o \geq 2$ and vanishing scalar component of the germ of harmonic curvature at o).

Proof. For each object \mathcal{D} of **Germ**, note that $\dim \text{sym } \mathcal{D} \leq 21$ so that we may, and do, choose a connected, simply-connected Lie group $K_{\mathcal{D}}$ together with a Lie algebra isomorphism $\mathfrak{k}_{\mathcal{D}} \rightarrow \text{sym } \mathcal{D}$. Then, let $L_{\mathcal{D}} \subset K_{\mathcal{D}}$ be the subgroup whose Lie algebra corresponds to the isotropy algebra in $\text{sym } \mathcal{D}$, and such that $K_{\mathcal{D}}/L_{\mathcal{D}}$ is simply-connected. The infinitesimal action of $\text{sym } \mathcal{D}$ on a small neighbourhood of $o \in \mathbb{R}^8$ induces a germ of a local diffeomorphism from $K_{\mathcal{D}}/L_{\mathcal{D}}$ to \mathbb{R}^8 , mapping the origin to o , and pulling back \mathcal{D} to the germ of a $K_{\mathcal{D}}$ -equivariant distribution $\tilde{\mathcal{D}}$ on $K_{\mathcal{D}}/L_{\mathcal{D}}$. It is then straightforward to check that mapping \mathcal{D} to $(K_{\mathcal{D}}/L_{\mathcal{D}}, \tilde{\mathcal{D}})$ extends to a functor $F_0 : \mathbf{Germ} \rightarrow \mathbf{Model}$.

In the opposite direction, we choose for each (M, \mathcal{D}, m) in **Model** a diffeomorphism $f : U \rightarrow \mathbb{R}^8$ from an open neighbourhood U of $m \in M$ to an open subset of \mathbb{R}^8 such that $f(m) = o$. Then $\bar{\mathcal{D}} = [(f(U), f_*\mathcal{D})]$ defines an object of **Germ**, and it is again straightforward to check that it extends to a functor $G_0 : \mathbf{Model} \rightarrow \mathbf{Germ}$.

Let $I : \mathbf{Model} \rightarrow \mathbf{FracModel}$ be the canonical embedding. Since **Germ** is a groupoid, it follows that G_0 factors uniquely through I so that $G_0 = GI$. Set $F = IF_0$. We now need too check that the pair G, F forms an equivalence of categories. First, we have a natural isomorphism $\text{id} \rightarrow G_0F_0 = GF$ whose component at \mathcal{D} is the germ of a local diffeomorphism $\mathbb{R}^8 \rightarrow \mathbb{R}^8$ arising as the composite of germs of diffeomorphisms $\mathbb{R}^8 \rightarrow K_{\mathcal{D}}/L_{\mathcal{D}} \rightarrow \mathbb{R}^8$ arising in the definition of F_0, G_0 . On the opposite side, there is a natural isomorphism $\text{id} \rightarrow FG$ whose component at (M, \mathcal{D}, m) is given by the diagram $M \leftarrow U \rightarrow K_{\bar{\mathcal{D}}}/L_{\bar{\mathcal{D}}}$ where the rightmost map is a representative of the composite of germs at m of local diffeomorphisms $M \rightarrow \mathbb{R}^8 \rightarrow K_{\bar{\mathcal{D}}}/L_{\bar{\mathcal{D}}}$ arising in the definition of G_0, F_0 . \square

As a side remark, note that the category of parabolic geometries (of a given type) and local equivalences is enriched over the category of finite-dimensional smooth manifolds. Using Proposition 1, the same holds for **Model** and $\mathbf{FracModel} \approx \mathbf{Germ}$, whence in particular $\text{Aut } \mathcal{D}$ for \mathcal{D} in **Germ** is naturally a Lie group: the isotropy group at $o \in \mathbb{R}^8$. It follows that the isotropy condition appearing in our original classification problem corresponds $\dim \text{Aut } \mathcal{D} \geq 2$. We shall not pursue this interpretation.

2. FROM DISTRIBUTIONS TO DEFORMATIONS

2.1. Introduction. The central idea of our approach to the problem of classification of homogeneous models is to reformulate it in terms of the deformation theory of filtered Lie algebras. The way we present it in this section applies to arbitrary strongly regular distributions whose symbol (\mathfrak{g}_- in our case) has a finite-dimensional Tanaka prolongation (\mathfrak{g} in our case): indeed, these are the only properties of $(\mathfrak{g}_-, \mathfrak{g})$ we shall refer to in what follows.

Our first step will be a pretty standard trick: replace the property of ‘there exists a transitive symmetry algebra’ with the *datum* of such algebra. That is, we will consider germs of Monge C_3 distributions \mathcal{D} together with an explicit transitive algebra of germs of vector fields $\mathfrak{k} \subset \text{sym } \mathcal{D}$. Now, classifying pairs $(\mathcal{D}, \mathfrak{k})$ up to equivalence yields a partially ordered set rather than a set, where the order relation reflects inclusions between the algebras. Following the well-known dictum ‘a groupoid is the categorification of a set; a category is the categorification of a poset’, we shall organise the corresponding equivalence problem into a category rather than a groupoid.

2.2. Symmetries and the symbol. As a preliminary step, let us recall a basic fact about local symmetry algebras of distributions (see e.g. [4]). Let **FLA**, resp. **GLA**, denote the category of finite-dimensional filtered, resp. graded, Lie algebras and filtration-, resp. grading-preserving homomorphisms. The associated graded construction gives a functor $\text{gr} : \mathbf{FLA} \rightarrow \mathbf{GLA}$. Denote by **Symbol** $\subset \mathbf{GLA}$ the subcategory whose objects are negatively graded algebras, generated in degree -1 , with *finite-dimensional Tanaka prolongation*, and whose morphisms are graded Lie algebra isomorphisms. We then have the Tanaka prolongation functor

$$\text{Pr} : \mathbf{Symbol} \rightarrow \mathbf{GLA}$$

together with a natural monomorphism $\text{id} \rightarrow \text{Pr}$ (of functors $\mathbf{Symbol} \rightarrow \mathbf{GLA}$).

Given a germ of a bracket-generating distribution \mathcal{D} at $o \in \mathbb{R}^8$, recall that \mathcal{D} induces a germ of a filtration on the tangent bundle, and thus in particular a filtration $T_o^\bullet \mathbb{R}^8$ on the tangent space at o . It is the weakest filtration compatible with the Lie bracket of vector fields, and such that its degree -1 , resp. 0 , sub-bundle is precisely \mathcal{D} , resp. zero. It follows that the Lie bracket turns the associated graded $\text{gr } T_o \mathbb{R}^8$ into a graded nilpotent Lie algebra $\sigma_\bullet(\mathcal{D})$. This is easily seen to produce a functor

$$\sigma : \mathbf{Germ} \rightarrow \mathbf{Symbol}.$$

Lemma 1. *The assignment $\mathcal{D} \mapsto \text{sym } \mathcal{D}$ extends naturally to a functor $\text{sym} : \mathbf{Germ} \rightarrow \mathbf{FLA}$ together with a natural monomorphism $\sigma \rightarrow \text{gr sym} \rightarrow \text{Pr } \sigma$ of functors $\mathbf{Germ} \rightarrow \mathbf{GLA}$.*

Proof. Let \mathcal{D} be an object of \mathbf{Germ} . We first need to exhibit the filtration on $\text{sym } \mathcal{D}$. We set $\text{sym}^i \mathcal{D} = \text{ev}_o^{-1} T_o^i \mathbb{R}^8$ for $i < 0$, and then let

$$\text{sym}^i \mathcal{D} = \{X \in \text{sym } \mathcal{D} \mid \text{ev}_o[\mathcal{D}, [\dots[\mathcal{D}, X]]] = 0 \text{ (} i \text{ copies)}\}$$

for $i \geq 0$. It is straightforward to check functoriality. The inclusion $\sigma(\mathcal{D}) \rightarrow \text{gr sym } \mathcal{D}$ is clear by construction: indeed, the symbol is identified with the negative part of $\text{gr sym } \mathcal{D}$. Then the homomorphism $\text{gr sym } \mathcal{D} \rightarrow \text{Pr } \sigma(\mathcal{D})$ arises from the universal property of Tanaka prolongation with respect to graded Lie algebras extending the symbol in non-negative degrees. Naturality of the two maps is again straightforward to check.

It remains to verify injectivity of $\text{gr}_i \text{sym } \mathcal{D} \rightarrow \text{Pr } \sigma(\mathcal{D})_i$ for all i . Again, for $i < 0$ this follows directly from the construction. Letting \mathcal{D}_o be the fibre of \mathcal{D} at $o \in \mathbb{R}^8$, we then have for each $i \geq 0$ a commutative diagram

$$\begin{array}{ccccc} \text{sym}^i \mathcal{D} & \xrightarrow{\quad} & \text{gr}_i \text{sym } \mathcal{D} & \xrightarrow{\quad} & \text{Pr } \sigma(\mathcal{D})_i \\ & \searrow & \downarrow & & \downarrow \\ & & \text{Hom}(\bigotimes^{i+1} \mathcal{D}_o, \mathcal{D}_o) & \xrightarrow{\cong} & \text{Hom}(\bigotimes^{i+1} \sigma_{-1}(\mathcal{D}), \sigma_{-1}(\mathcal{D})) \end{array}$$

where the vertical arrows are induced by the Lie bracket of vector fields (on the left), resp. the adjoint representation (on the right). By the defining properties of the Tanaka prolongation, the right vertical arrow is injective, so that it is enough to check that the left one is. Suppose thus $X \in \text{sym}^i \mathcal{D}$ is such that $\bar{X} \in \text{gr}_i \text{sym } \mathcal{D}$ is mapped to $0 : \bigotimes^{i+1} \mathcal{D}_o \rightarrow \mathcal{D}_o$; then $X \in \text{sym}^{i+1} \mathcal{D}$ by definition of the filtration, whence $\bar{X} = 0$. \square

2.3. Distributions and deformations. As we have indicated in the introductory subsection, we shall proceed by adding a transitive Lie algebra of symmetries (not necessarily the maximal one) as a datum in the classification problem.

Definition 4. **GermSym** is the category whose objects are pairs $(\mathcal{D}, \mathfrak{k})$ such that \mathcal{D} is an object of \mathbf{Germ} , and $\mathfrak{k} \subset \text{sym } \mathcal{D}$ is a Lie subalgebra such that $\text{ev}_o : \mathfrak{k} \rightarrow T_o \mathbb{R}^8$ is surjective; its morphisms from $(\mathcal{D}, \mathfrak{k})$ to $(\mathcal{D}', \mathfrak{k}')$ are those morphisms from \mathcal{D} to \mathcal{D}' in \mathbf{Germ} whose underlying germ of a diffeomorphism $\mathbb{R}^8 \rightarrow \mathbb{R}^8$ maps \mathfrak{k} into \mathfrak{k}' .

Lemma 2. *The obvious forgetful functor $\mathbf{GermSym} \rightarrow \mathbf{Germ}$ admits a full and faithful right adjoint.*

Proof. The right adjoint sends \mathcal{D} in \mathbf{Germ} to $(\mathcal{D}, \text{sym } \mathcal{D})$ in $\mathbf{GermSym}$, and its action on morphisms is given by functoriality of sym . Indeed, morphisms $(\mathcal{D}, \mathfrak{k}) \rightarrow (\mathcal{D}', \text{sym } \mathcal{D}')$ in $\mathbf{GermSym}$ are the same as morphisms $\mathcal{D} \rightarrow \mathcal{D}'$ in \mathbf{Germ} . Taking $\mathfrak{k} = \text{sym } \mathcal{D}$, we also find that this right adjoint is full and faithful. \square

The adjunction $\mathbf{Germ} \rightleftarrows \mathbf{GermSym}$ turns \mathbf{Germ} into a *reflexive* subcategory of $\mathbf{GermSym}$. In particular, we have the composite functor $S : \mathbf{GermSym} \rightarrow \mathbf{GermSym}$, sending $(\mathcal{D}, \mathfrak{k})$ to $(\mathcal{D}, \text{sym } \mathcal{D})$. Then \mathbf{Germ} is precisely the full subcategory consisting of objects preserved by S ; obviously that is merely an elaborate way to say $\mathfrak{k} = \text{sym } \mathcal{D}$. This way we have embedded the groupoid controlling our classification problem into a larger (but still small!) category. Our strategy now will be to describe the set of isomorphism classes of objects of $\mathbf{GermSym}$, and only later to check which ones are preserved by S . The main reason for admitting this seemingly spurious wealth of objects is that we may now completely forget about distributions!

Lemma 3. *The obvious forgetful functor $\mathbf{GermSym} \rightarrow \mathbf{FLA}$ is a full and faithful embedding. Its essential image is the full subcategory of filtered Lie algebras \mathfrak{k} admitting a graded Lie algebra monomorphism $\text{gr } \mathfrak{k} \rightarrow \mathfrak{g}$ whose image contains \mathfrak{g}_- .*

Proof. Let $\mathbf{C} \subset \mathbf{FLA}$ denote the full subcategory described in the statement of the Lemma. Clearly, the forgetful functor factors through $A : \mathbf{GermSym} \rightarrow \mathbf{C}$. We shall construct an essential inverse $B : \mathbf{C} \rightarrow \mathbf{GermSym}$. First, for each \mathfrak{k} in \mathbf{C} choose a connected, simply connected Lie group K together with an identification of its Lie algebra with \mathfrak{k} . Let $K^0 \subset K$ be the subgroup with Lie sub-algebra \mathfrak{k}^0 such that K/K^0 is simply-connected. Let $\mathcal{D}_{\mathfrak{k}}$ be the K -invariant distribution on K/K^0 corresponding to $\mathfrak{k}^{-1}/\mathfrak{k}^0$. Since \mathfrak{k} is in \mathbf{C} , it follows that $(K/K^0, \mathcal{D}_{\mathfrak{k}})$ is an object of \mathbf{Model} . Choose a germ $f_{\mathfrak{k}}$ of a diffeomorphism $K/K^0 \rightarrow \mathbb{R}^8$ sending the origin to o (cf. the construction of G_0 in the proof of Proposition 2). Finally let $B(\mathfrak{k}) = (f_{\mathfrak{k}*} \mathcal{D}_{\mathfrak{k}}, f_{\mathfrak{k}*} \mathfrak{k})$ where \mathfrak{k} is viewed as a Lie algebra of vector fields on K/K^0 . It is straightforward to check that $\mathfrak{k} \rightarrow B(\mathfrak{k})$ extends to a functor. Now, $AB \simeq \text{id}_{\mathbf{C}}$ by construction. On the other hand, the natural isomorphism $BA \simeq \text{id}_{\mathbf{GermSym}}$ is given on $(\mathcal{D}, \mathfrak{k})$ by the germ of a diffeomorphism $K/K^0 \rightarrow \mathbb{R}^8$ integrating the \mathfrak{k} -action on both sides. \square

We apply the well-known trick once again and replace the property ‘there exists...’ with explicit data.

Definition 5. **Def** is the category whose objects are pairs (\mathfrak{k}, ι) where \mathfrak{k} is a filtered Lie algebra, and $\iota : \text{gr } \mathfrak{k} \rightarrow \mathfrak{g}$ is a graded Lie algebra monomorphism such that $\mathfrak{g}_- \subset \iota(\text{gr } \mathfrak{k})$. Its morphisms from (\mathfrak{k}, ι) to (\mathfrak{k}', ι') are pairs (φ, g) where $\varphi : \mathfrak{k} \rightarrow \mathfrak{k}'$ is a filtered Lie algebra homomorphism, and $g \in G_0$ an element such that $\iota' \circ \text{gr } \varphi = \iota \circ \text{Ad}_g$.

Lemma 4. *Let (\mathfrak{k}, ι) and (\mathfrak{k}', ι') be objects of **Def**. Then for each filtered Lie algebra homomorphism $\varphi : \mathfrak{k} \rightarrow \mathfrak{k}'$ there exists a unique $g \in G_0$ such that (φ, g) is a morphism $(\mathfrak{k}, \iota) \rightarrow (\mathfrak{k}', \iota')$ in **Def**.*

Proof. Note that $\text{gr } \varphi$ defines a graded Lie algebra automorphism

$$\mathfrak{g}_- \xrightarrow{\iota'^{-1}} \mathfrak{k}'_- \xrightarrow{\text{gr } \varphi} \mathfrak{k}'_- \xrightarrow{\iota'} \mathfrak{g}_-$$

and thus an element $g \in G_0$. By the universal property of Tanaka prolongation, we then have $\iota' \circ \text{gr } \varphi = \iota \circ \text{Ad}_g$. On the other hand, every element $g \in G_0$ satisfying the latter equation induces the same automorphism of \mathfrak{g}_- , whence uniqueness. \square

Lemma 5. *The obvious forgetful functor $\mathbf{Def} \rightarrow \mathbf{FLA}$ is a full and faithful embedding onto the essential image of the other obvious forgetful functor $\mathbf{GermSym} \rightarrow \mathbf{FLA}$. As a consequence, there is an equivalence of categories between \mathbf{Def} and $\mathbf{GermSym}$.*

Proof. Once again we use the notation $\mathbf{C} \subset \mathbf{FLA}$ for the full subcategory in question. The forgetful functor factors through $A : \mathbf{Def} \rightarrow \mathbf{C}$ by definition. Its essential inverse $B : \mathbf{C} \rightarrow \mathbf{Def}$ is constructed by choosing for each \mathfrak{k} in \mathbf{C} a graded Lie algebra monomorphism $\iota_{\mathfrak{k}} : \text{gr } \mathfrak{k} \rightarrow \mathfrak{g}$ such that $B(\mathfrak{k}) = (\mathfrak{k}, \iota_{\mathfrak{k}})$ is an object of \mathbf{Def} . Given a filtered homomorphism $\varphi : \mathfrak{k} \rightarrow \mathfrak{k}'$ in \mathbf{C} , let $g_{\varphi} \in G_0$ be the unique element such that (φ, g_{φ}) is a morphism $B(\mathfrak{k}) \rightarrow B(\mathfrak{k}')$ in \mathbf{Def} (cf. Lemma 4). Set $B(\varphi) = (\varphi, g_{\varphi})$. The natural isomorphisms $AB \simeq \text{id}_{\mathbf{C}}$ and $BA \simeq \text{id}_{\mathbf{Def}}$ are tautological: the component of the former at \mathfrak{k} is $\text{id}_{\mathfrak{k}}$ as a morphism in \mathbf{C} , while the component of the latter at (\mathfrak{k}, ι) is $\text{id}_{\mathfrak{k}}$ as a morphism to $(\mathfrak{k}, \iota_{\mathfrak{k}})$ in \mathbf{Def} . \square

Our last step in this sub-section is to switch focus from the filtered Lie algebra \mathfrak{k} to the image of ι , a graded subalgebra $\underline{\mathfrak{k}} \subset \mathfrak{g}$ (we will use this notational convention throughout the paper).

Definition 6. **Sub** is the category whose objects are graded sub-algebras of \mathfrak{g} containing \mathfrak{g}_- , and whose morphisms from $\underline{\mathfrak{k}}$ to $\underline{\mathfrak{k}}'$ are elements $g \in G_0$ such that $\text{Ad}_g \underline{\mathfrak{k}} \subset \underline{\mathfrak{k}}'$.

The reason for including an element of G_0 explicitly in the definition of morphisms in \mathbf{Def} is that we now have a functor $\mathbf{Def} \rightarrow \mathbf{Sub}$ sending (\mathfrak{k}, ι) to $\iota(\text{gr } \mathfrak{k})$ and (φ, g) to g . We thus view \mathbf{Def} as a category over \mathbf{Sub} . Recall that the *fibre* $\mathbf{Def}_{\underline{\mathfrak{k}}}$ of \mathbf{Def} over an object $\underline{\mathfrak{k}}$ of \mathbf{Sub} is the sub-category of \mathbf{Def} whose objects are mapped to $\underline{\mathfrak{k}}$ in \mathbf{Sub} , and whose morphisms are mapped to $\text{id}_{\underline{\mathfrak{k}}}$ in \mathbf{Sub} . In particular, in our case $\mathbf{Def}_{\underline{\mathfrak{k}}}$ is a large groupoid: indeed, a filtered Lie algebra homomorphism $\varphi : \mathfrak{k} \rightarrow \mathfrak{k}'$ is an isomorphism if (and only if) $\text{gr } \varphi$ is. Now, given (\mathfrak{k}, ι) in $\mathbf{Def}_{\underline{\mathfrak{k}}}$, we may view \mathfrak{k} as a *filtered deformation* of $\underline{\mathfrak{k}}$, trivial under passage to the associated graded. We shall thus use deformation theory to study the fibres of $\mathbf{Def} \rightarrow \mathbf{Sub}$.

2.4. The formalism of DGLAs. Deligne's principle states that every reasonably well-behaved deformation problem (in char. 0) is controlled by a *differential Graded Lie algebra* (abbreviated DGLA). That is, the equivalence problem for deformations of an algebraic or geometric object (in our case encoded in the large groupoid $\mathbf{Def}_{\underline{\mathfrak{k}}}$) may be replaced with a standard equivalence problem associated with a DLGA. One may then consider the question of existence of a 'deformation space' abstracting from the contingencies of the original objects (in our case, this at the very least allows us to avoid unnecessarily cluttered notation). This subsection reviews the relevant notions and constructions without any reference to the remaining parts of the paper. We shall resume the main narrative in the next subsection. In part, we adapt the presentation of [5].

It is difficult to avoid a slight terminological inconsistency in the use of the word 'graded Lie algebra'. Thus far, it denoted a Lie algebra together with a grading compatible with the Lie bracket. On the other hand, in the context of DGLA, the term denotes essentially what is called a Lie super-algebra. We shall use a

capitalised ‘Graded’ for the latter meaning; furthermore, Graded degree will appear as an upper index (as opposed to graded degree, appearing as a lower index). That is, a Graded vector space is $V = \bigoplus_p V^p$; given a homogeneous element $v \in V^p$, we set $|v| = p$.

Definition 7. A DGLA is a Graded vector space $\mathfrak{L} = \bigoplus_p \mathfrak{L}^p$ together with:

- (1) a bracket $[\cdot, \cdot] : \mathfrak{L} \otimes \mathfrak{L} \rightarrow \mathfrak{L}$ of degree 0,
- (2) a differential $d : \mathfrak{L} \rightarrow \mathfrak{L}$ of degree 1, $d^2 = 0$,

satisfying:

- (1) Graded skewness $[x, y] = (-1)^{|x||y|+1}[y, x]$,
- (2) Graded Jacobi identity $(-1)^{|x||z|}[x, [y, z]] + \text{cycl.} = 0$,
- (3) Graded Leibniz rule $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$

for homogeneous elements $x, y, z \in \mathfrak{L}$.

Note that a DGLA is in particular a cochain complex. We define the cycles $Z^p(\mathfrak{L})$, boundaries $B^p(\mathfrak{L})$ and cohomology $H^p(\mathfrak{L}) = Z^p(\mathfrak{L})/B^p(\mathfrak{L})$ in the usual way. Observe also that \mathfrak{L}^0 is a (usual) Lie algebra acting on \mathfrak{L} by Graded derivations of degree 0. If this adjoint action is nilpotent, we consider the connected, simply connected Lie group $\exp \mathfrak{L}^0$ and its induced ‘adjoint’ action by automorphisms of \mathfrak{L} , denoted Ad . We may then identify $\exp \mathfrak{L}^0$ with \mathfrak{L}^0 as a manifold, so that the exponential map becomes the identity, while the multiplication and inverse, as well as Ad , are given by polynomial maps.

The ‘standard’ equivalence problem associated with a DGLA is expressed by the following notions.

Definition 8. Let \mathfrak{L} be a finite-dimensional DGLA.

- (1) An element $x \in \mathfrak{L}^1$ is *Maurer-Cartan* if

$$dx + \frac{1}{2}[x, x] = 0.$$

The algebraic subset of Maurer-Cartan elements is denoted $\text{MC}(\mathfrak{L}) \subset \mathfrak{L}^1$.

- (2) The *infinitesimal gauge action* is an affine action of \mathfrak{L}^0 on \mathfrak{L}^1 defined by

$$* : \mathfrak{L}^0 \times \mathfrak{L}^1 \rightarrow \mathfrak{L}^1, \quad y * x = [y, x] - dy.$$

If the adjoint action of \mathfrak{L}^0 on \mathfrak{L} is nilpotent, we exponentiate the above to the *gauge action* of $\exp \mathfrak{L}^0$ on \mathfrak{L}^1 . Together with this action, $\exp \mathfrak{L}^0$ is the *gauge group* of \mathfrak{L} .

It is a simple exercise to check that the gauge action of $\exp \mathfrak{L}^0$ preserves $\text{MC}(\mathfrak{L}) \subset \mathfrak{L}^1$. We may thus consider the question of gauge-equivalence of Maurer-Cartan elements.

Definition 9. Under the above assumptions, the *gauge action groupoid* $\text{MC}(\mathfrak{L}) // \exp \mathfrak{L}^0$ has $\text{MC}(\mathfrak{L})$ as the set of objects, and

$$\text{Hom}(x, y) = \{u \in \exp \mathfrak{L}^0 \mid u * x = y\}$$

as homsets for $x, y \in \text{MC}(\mathfrak{L})$.

Observe that the gauge action is affine. In fact, one may consider an extension $\mathfrak{L} \oplus \langle d \rangle$ of the original DGLA obtained by formally adjoining an element d in Graded degree 1 with the obvious relations $dd = 0$, $[d, x] = dx$. The adjoint action of $\exp \mathfrak{L}^0$ extends naturally to an action on the extended DGLA, preserving the

affine subspace $d + \mathfrak{L}$. Then, identifying \mathfrak{L} with the affine subspace $d + \mathfrak{L}$, one checks that the gauge action on \mathfrak{L} corresponds to the naturally extended adjoint action, restricted to $d + \mathfrak{L}$. In symbols, $\text{Ad}_u(d + x) = d + u * x$.

An ideal solution to the classification problem would be to construct the ‘deformation space’ $\text{MC}(\mathfrak{L})/\exp \mathfrak{L}^0$ as a manifold or variety. In general, this only possible *formally* around the trivial deformation represented by $0 \in \mathfrak{L}^1$, and furthermore up to some residual equivalence. The ‘optimal’ formal deformation space is then a so-called miniversal family, characterised by the property that its tangent space at the origin is identified with $H^1(\mathfrak{L})$: the true space of first-order deformations. The *Kuranishi family* is a miniversal family realised as an often singular formal subvariety in $H^1(\mathfrak{L})$. In our case it will turn out that the construction may be carried out *globally*, producing an actual subvariety of $H^1(\mathfrak{L})$, which will furthermore turn out to be the actual deformation space, i.e. a *universal* family.

The features of our deformation problem that allow for a global construction are captured abstractly in the following notion.

Definition 10. A *graded nilpotent DGLA* is a finite-dimensional DGLA $\mathfrak{L} = \bigoplus_p \mathfrak{L}^p$ together with a grading $\mathfrak{L}^p = \bigoplus_{i>0} \mathfrak{L}_i^p$ in *positive* degrees such that both the bracket and the differential are of graded degree zero.

Note that we have added a ‘lower-case’ grading to the data. In particular, $\mathfrak{L}^0 = \bigoplus_{i>0} \mathfrak{L}_i^0$ becomes a (positively) graded nilpotent Lie algebra, with a compatible action on the graded vector space $\mathfrak{L} = \bigoplus_{i>0} \mathfrak{L}_i$. The next feature of our particular deformation problem is the very strong vanishing condition $H^0(\mathfrak{L}) = 0$, implying that the trivial deformation has a trivial stabiliser in the gauge group. As we shall see, this does in fact imply that the gauge group acts freely on $\text{MC}(\mathfrak{L})$.

Proposition 3. *Let \mathfrak{L} be a graded nilpotent DGLA with $H^0(\mathfrak{L}) = 0$. Then there is an algebraic subset $M \subset H^1(\mathfrak{L})$ together with an algebraic map $\pi : \text{MC}(\mathfrak{L}) \rightarrow M$, an algebraically trivial principal bundle for the gauge group. As a consequence, the gauge action groupoid $\text{MC}(\mathfrak{L}) // \exp \mathfrak{L}^0$ is equivalent to the discrete groupoid over M .*

We will refer to such M , together with an algebraic section $\xi : M \rightarrow \text{MC}(\mathfrak{L})$, as a (global, universal) Kuranishi family.

Proof. Choose a splitting

$$\mathfrak{L} = Z(\mathfrak{L}) \oplus C = B(\mathfrak{L}) \oplus H(\mathfrak{L}) \oplus C$$

on the level of bigraded (i.e. graded Graded) vector spaces. Note that the restriction $d|_C : C \rightarrow B(\mathfrak{L})$ of the differential is invertible. Let $\delta : \mathfrak{L} \rightarrow \mathfrak{L}$ be the map of Graded degree -1 (and graded degree 0) defined as $d|_C^{-1} : B(\mathfrak{L}) \rightarrow C$ pre-composed with projection $\mathfrak{L} \rightarrow B(\mathfrak{L})$ and post-composed with inclusion $C \rightarrow \mathfrak{L}$. Note that $\delta^2 = 0$ and $d\delta$ is the projection onto $B(\mathfrak{L})$ while δd is the projection onto C . Consider the algebraic map

$$\Phi : \mathfrak{L}^1 \rightarrow \mathfrak{L}^1, \quad \Phi(x) = x + \frac{1}{2}\delta[x, x].$$

We claim that it possesses the following properties:

- (1) Φ admits an algebraic inverse,
- (2) Φ identifies $\text{MC}(\mathfrak{L})$ with the set

$$\{x \in Z^1(\mathfrak{L}) \mid [\Phi^{-1}x, \Phi^{-1}x] \in B^2(\mathfrak{L})\},$$

(3) Φ identifies $\text{MC}(\mathfrak{L}) \cap (H^1(\mathfrak{L}) \oplus C^1)$ with $\Phi(\text{MC}(\mathfrak{L})) \cap H^1(\mathfrak{L})$.

For (1), observe that the additional nilpotent grading on \mathfrak{L} allows one to solve the equation $x + \frac{1}{2}\delta[x, x] = y$ degree by degree. For (2), observe that given $x \in \mathfrak{L}^1$ such that $[x, x] \in B^2(\mathfrak{L})$, we have $d\Phi(x) = dx + \frac{1}{2}[x, x]$ indentially. For (3), observe that given $x \in \text{MC}(\mathfrak{L})$ we have $d\Phi(x) = 0$ as well as $\delta\Phi(x) = \delta x$. We now define M to be the zero-locus of the quadratic map

$$H^1(\mathfrak{L}) \hookrightarrow \mathfrak{L}^1 \xrightarrow{\Phi^{-1}} \mathfrak{L}^1 \xrightarrow{[\cdot, \cdot]} \mathfrak{L}^2 \rightarrow H^2(\mathfrak{L}) \oplus C^2$$

so that $[\Phi^{-1}(x), \Phi^{-1}(x)] \in B^2(\mathfrak{L})$ for all $x \in M$. It follows that there is a pullback diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad} & \mathfrak{L}^1 \\ \xi \downarrow & & \downarrow \Phi^{-1} \\ \text{MC}(\mathfrak{L}) \cap (H^1(\mathfrak{L}) \oplus C^1) & \longrightarrow & \mathfrak{L}^1. \end{array}$$

identifying M with the zero-locus of δ in $\text{MC}(\mathfrak{L})$.

We now ask whether this zero-locus intersects all gauge orbits. Given $x \in \text{MC}(\mathfrak{L})$, define the function

$$\Psi_x : C^0 \rightarrow C^0, \quad \Psi(y) = \delta(e^y * x).$$

Again using the nilpotent grading on \mathfrak{L} and solving the equation $\delta(e^y * x) = z$ degree by degree, one finds that Ψ_x admits an algebraic inverse, and furthermore the latter depends algebraically on x (the degree i graded component of this equation is of the form $y_i = \delta dy_i = \delta(\dots)$ where (\dots) involves only y_j , $j < i$). Thus, the zero-locus of the map

$$C^0 \times \text{MC}(\mathfrak{L}) \rightarrow C^0, \quad (y, x) \mapsto \Psi_x y$$

is precisely the graph of an algebraic map $\eta : \text{MC}(\mathfrak{L}) \rightarrow C^0$ such that

$$e^{\eta(x)} * x \in H^1(\mathfrak{L}) \oplus C^1(\mathfrak{L})$$

for all $x \in \text{MC}(\mathfrak{L})$. In particular, we may define the projection

$$\pi : \text{MC}(\mathfrak{L}) \rightarrow M, \quad \pi(x) = \Phi(e^{\eta(x)} * x)$$

so that $\pi \circ \xi = \text{id}_M$.

Of course $C^0 = \mathfrak{L}^0$ by the hypothesis $H^0(\mathfrak{L}) = 0$. It thus follows that the action map

$$\exp \mathfrak{L}^0 \times M \times \text{MC}(\mathfrak{L}), \quad (e^y, m) \mapsto e^y * \xi(m)$$

admits an algebraic inverse sending $x \in \text{MC}(\mathfrak{L})$ to $(e^{-\eta(x)}, \pi(x)) \in \exp \mathfrak{L}^0 \times M$. Hence, $\pi : \text{MC}(\mathfrak{L}) \rightarrow M$ is a trivial principal bundle for the gauge action of $\exp \mathfrak{L}^0$. Finally, π induces a homomorphism of groupoids from $\text{MC}(\mathfrak{L}) // \exp \mathfrak{L}^0$ to M (discrete), sending the morphism $x \rightarrow x'$ given by $u \in \exp \mathfrak{L}^0$ to the identity morphism of $\pi(x) = \pi(x')$. By freeness of the action of the structure group of a principal bundle, it follows that the automorphism group of an object in $\text{MC}(\mathfrak{L}) // \exp \mathfrak{L}^0$ is trivial and thus the induced maps on automorphism groups are isomorphisms. Thus, the above homomorphism is an equivalence of groupoids. \square

We remark that even without the condition $H^0(\mathfrak{L}) = 0$ we may carry out the construction of $M \subset H^1(\mathfrak{L})$ together with the quotient map $\pi : \text{MC}(\mathfrak{L}) \rightarrow M$ and a section ξ identifying M with $\text{MC}(\mathfrak{L}) \cap (H^1(\mathfrak{L}) + C^1)$. However, if $H^0(\mathfrak{L})$ is nontrivial, π is no longer a principal bundle. Nevertheless, M may still be a universal family,

i.e. the true quotient $\mathrm{MC}(\mathfrak{L})/\exp \mathfrak{L}^0$, as long as the gauge subgroups stabilising all Maurer-Cartan elements are of the same dimension $\dim H^0(\mathfrak{L})$.

2.5. Back to deformations: fibres. We now return to the previous setting. Our aim is to describe the fibre $\mathbf{Def}_{\mathfrak{L}}$ as equivalent to the action groupoid $\mathrm{MC}(\mathfrak{L})/\exp \mathfrak{L}^0$ for a suitable DGLA controlling filtered deformations of \mathfrak{L} with a trivial associated graded. This will then allow us to pass to the Kuranishi space M , as a discrete groupoid (as long as the zeroth cohomology of \mathfrak{L} vanishes). In particular, the points of $M \subset H^1(\mathfrak{L})$ will be in bijection with isomorphism classes of objects of $\mathbf{Def}_{\mathfrak{L}}$. We refer to [3] for the background on deformations of Lie algebras.

We begin by recalling some further Graded notions, following the conventions introduced in the preceding subsection.

Definition 11.

- (1) A Graded-commutative algebra is a Graded vector space $A = \bigoplus_p A^p$ together with an associative bilinear operation \cdot of degree 0 such that

$$xy = (-1)^{|x||y|}yx$$

for homogeneous elements $x, y \in A$.

- (2) A Graded derivation of A of degree r is a degree r linear map $\delta : A \rightarrow A$ satisfying the graded Leibniz identity

$$\delta(xy) = (\delta x)y + (-1)x(\delta y).$$

- (3) A Graded Lie algebra is a Graded vector space $\mathfrak{L} = \bigoplus_p \mathfrak{L}^p$ together with a bracket $[\cdot, \cdot] : \mathfrak{L} \otimes \mathfrak{L}$ of degree 0 satisfying Graded skewness and Graded Leibniz rule (cf. Definition 7).
- (4) The Graded Lie algebra of derivations of a Graded-commutative algebra A is the Graded vector space $\mathrm{Der} A = \bigoplus_p \mathrm{Der}^p A$ where $\mathrm{Der}^p A$ consists of degree p Graded derivations of A , together with the bracket defined by

$$[\delta, \delta'] = \delta\delta' - (-1)^{|\delta||\delta'|}\delta'\delta$$

on homogeneous elements, and extended by bilinearity.

Let us now fix a graded Lie algebra \mathfrak{L} . The exterior algebra $\Lambda^\bullet \mathfrak{L}^*$ is naturally a Graded-commutative algebra, and \mathfrak{L} embeds in $\mathrm{Der}^{-1} \Lambda^\bullet \mathfrak{L}^*$ via the evaluation map. Consider the cochain complex

$$C^\bullet(\mathfrak{L}, \mathfrak{L}) = \Lambda^\bullet \mathfrak{L}^* \otimes \mathfrak{L}$$

with differential d computing Lie algebra cohomology with coefficients in the adjoint representation. We define an injective map

$$i : \Lambda^\bullet \mathfrak{L}^* \otimes \mathfrak{L} \rightarrow \mathrm{Der} \Lambda^\bullet \mathfrak{L}^*$$

$$i_{\alpha \otimes X} \beta = \alpha \wedge \beta(X).$$

such that $i_{\alpha \otimes X}$ is a degree $|\alpha| - 1$ Graded derivation for homogeneous α . Let $\tilde{\mathfrak{L}}(\mathfrak{L})$ be the image of i . One checks that it is a Graded Lie sub-algebra of $\mathrm{Der} \Lambda^\bullet \mathfrak{L}^*$, and furthermore becomes a DGLA with the differential d . Explicitly, we have

$$\tilde{\mathfrak{L}}^p(\mathfrak{L}) \simeq C^{p+1}(\mathfrak{L}, \mathfrak{L}).$$

The bracket in $\mathrm{Der} \Lambda^\bullet \mathfrak{L}^*$, restricted to $\tilde{\mathfrak{L}}(\mathfrak{L})$ and viewed as a bilinear operation on $C^\bullet(\mathfrak{L}, \mathfrak{L})$ is usually referred to as the *Richardson-Nijenhuis bracket*.

Since \mathfrak{k} was originally a graded Lie algebra, it follows that $\tilde{\mathfrak{L}}(\mathfrak{k})$ carries additionally an induced grading $\tilde{\mathfrak{L}}(\mathfrak{k}) = \bigoplus_i \tilde{\mathfrak{L}}(\mathfrak{k})_i$. While $\tilde{\mathfrak{L}}(\mathfrak{k})$ controls all deformations of \mathfrak{k} , we need to restrict to a subalgebra corresponding to filtered deformations whose associated graded is trivial. That corresponds to taking the components of strictly positive degree: we define

$$\mathfrak{L}(\mathfrak{k}) = \bigoplus_{j>0} \tilde{\mathfrak{L}}(\mathfrak{k})_j.$$

Since d and the Richardson-Nijenhuis bracket are compatible with the grading induced from \mathfrak{k} , it follows that $\mathfrak{L}(\mathfrak{k})$ is a *graded nilpotent sub-DGLA*. We leave it to the reader to convince themselves that the assignment $\mathfrak{k} \mapsto \mathfrak{L}(\mathfrak{k})$ extends to a functor

$$\mathfrak{L} : \mathbf{GLA} \rightarrow \mathbf{DGLA}$$

to the category of finite-dimensional DGLAs and their homomorphisms. Its relation to deformations may be finally revealed:

Proposition 4. *For each \mathfrak{k} in **Sub**, there is an equivalence of categories between $\mathbf{Def}_{\mathfrak{k}}$ and $\mathbf{MC}(\mathfrak{L}(\mathfrak{k})) // \exp \mathfrak{L}^0(\mathfrak{k})$.*

We need a pair of Lemmas.

Lemma 6. *Let \mathfrak{k} be a graded Lie algebra, and $\phi \in \mathfrak{L}^1(\mathfrak{k})$. Set $\mathfrak{k}_{\phi} = \mathfrak{k}$ as filtered vector spaces. Viewing ϕ as a map $\Lambda^2 \mathfrak{k}^* \rightarrow \mathfrak{k}$, define a deformed bracket on \mathfrak{k}_{ϕ} by*

$$[X, Y]_{\phi} = [X, Y] + \phi(X, Y).$$

Then $[-, -]_{\phi}$ satisfies the Jacobi identity if and only if ϕ is Maurer-Cartan. If that is the case, \mathfrak{k}_{ϕ} with $[\cdot, \cdot]_{\phi}$ is a filtered Lie algebra such that the tautological map $\mathfrak{gr} \mathfrak{k}_{\phi} \rightarrow \mathfrak{k}$ is an isomorphism of graded Lie algebras.

Proof. We compute:

$$\begin{aligned} [[X, Y]_{\phi}, Z]_{\phi} &= [[X, Y], Z] + [\phi(X, Y), Z] + \phi([X, Y], Z) \\ &\quad + 2\phi(\phi(X, Y), Z) \end{aligned}$$

so that Jacobi identity is equivalent to

$$[Z, \phi(X, Y)] - \phi([Z, X], Y) + 2\phi(Z, \phi(X, Y)) + \text{cycl.} = 0.$$

Now, the differential $d\phi$ is precisely

$$d\phi(X, Y, Z) = [Z, \phi(X, Y)] - \phi([Z, X], Y) + \text{cycl.}$$

On the other hand, the Richardson-Nijenhuis bracket is given by $i_{[\phi, \phi]} = 2i_{\phi}^2$ whence

$$[\phi, \phi](X, Y, Z) = 4\phi(\phi(X, Y), Z) + \text{cycl.}$$

and thus the Jacobi identity becomes $d\phi + \frac{1}{2}[\phi, \phi] = 0$. Now, if the above is satisfied, the deformed bracket is compatible with the filtration since ϕ is contained in the degree 0 filtered piece of $C^2(\mathfrak{k}, \mathfrak{k})$; furthermore, it induces the original bracket on the associated graded since ϕ is in fact contained in the degree 1 filtered piece. \square

Lemma 7. *Let $U \subset \mathbf{GL}(\mathfrak{k})$ be the unipotent subgroup consisting of filtration-preserving maps $u : \mathfrak{k} \rightarrow \mathfrak{k}$ with $\text{gr } u = \text{id}_{\mathfrak{k}}$, and let $\mathfrak{u} \subset \mathbf{End} \mathfrak{k}$ denote its Lie algebra. Then the embedding $C^{1,1}(\mathfrak{k}, \mathfrak{k}) \hookrightarrow \mathbf{End} \mathfrak{k}$ identifies $\mathfrak{L}^0(\mathfrak{k})$ with \mathfrak{u} and induces an identification of $\exp \mathfrak{L}^0(\mathfrak{k})$ with U such that, in the notation of Lemma 6,*

$$u : \mathfrak{k}_{\phi} \rightarrow \mathfrak{k}_{u*\phi}$$

is a filtered Lie algebra isomorphism for all $u \in U$, $\phi \in \mathbf{MC}(\mathfrak{L}^1(\mathfrak{k}))$.

Proof. Recall that the gauge action of $\exp \mathfrak{L}^0$ on $\mathfrak{L}^1(\mathfrak{k})$ may be identified with the restriction of its linear action on $\mathfrak{L}^1(\mathfrak{k}) \oplus \langle d \rangle$ to the linear subspace $d + \mathfrak{L}^1(\mathfrak{k})$. In the present case, we may embed $\mathfrak{L}(\mathfrak{k}) \oplus \langle d \rangle$ as a sub-DGLA of $\tilde{\mathfrak{L}}(\mathfrak{k})$, where the additional degree 1 element d is mapped to $d \text{id}$, with $\text{id} \in \tilde{\mathfrak{L}}^0(\mathfrak{k}) = \text{End } \mathfrak{k}$ being the identity map. We then have

$$d \text{id} + u * \phi = u(d \text{id} + \phi)$$

for all $u \in U$, $\phi \in \mathfrak{L}^1$. Now, computing

$$(d \text{id})(X, Y) = [X, \text{id } Y] - [Y, \text{id } X] - \text{id}[X, Y] = [X, Y]$$

we have that

$$[uX, uY]_{u*\phi} = u[X, Y]_\phi$$

as desired. \square

Proof of Proposition 4. The construction of Lemma 6 gives rise to a functor

$$F : \text{MC}(\mathfrak{L}(\mathfrak{k})) // \exp \mathfrak{L}^0(\mathfrak{k}) \rightarrow \mathbf{Def}_{\mathfrak{k}}$$

sending ϕ to \mathfrak{k}_ϕ together with the tautological map $\text{gr } \mathfrak{k}_\phi \rightarrow \mathfrak{k} \subset \mathfrak{g}$. An element $u \in U \simeq \exp \mathfrak{L}^0(\mathfrak{k})$, viewed as a morphism $\phi \rightarrow \phi'$ as in Lemma 7, is sent by F to itself viewed as a filtered map $\mathfrak{k}_\phi \rightarrow \mathfrak{k}_{\phi'}$. It is easy to see from this construction that F is full and faithful. Finally, to see that it is essentially surjective note that for any (\mathfrak{k}, ι) in $\mathbf{Def}_{\mathfrak{k}}$ we may choose an identification of \mathfrak{k} with $\text{gr } \mathfrak{k} \simeq \mathfrak{k}$ as vector spaces; then define an element $\phi : \Lambda^2 \mathfrak{k}^* \rightarrow \mathfrak{k}$ by $\phi(X, Y) = [X, Y]_{\mathfrak{k}} - [X, Y]_{\mathfrak{k}}$. By Lemma 6, ϕ is Maurer-Cartan and thus defines an object of $\text{MC}(\mathfrak{L}(\mathfrak{k})) // \exp \mathfrak{L}^0(\mathfrak{k})$. By construction, $F(\phi)$ is isomorphic to (\mathfrak{k}, ι) in $\mathbf{Def}_{\mathfrak{k}}$. \square

Corollary 1. *For each \mathfrak{k} in \mathbf{Sub} such that $H^0(\mathfrak{L}(\mathfrak{k})) = 0$ there is an equivalence of categories between $\mathbf{Def}_{\mathfrak{k}}$ and the discrete groupoid on $M_{\mathfrak{k}}$, where $(M_{\mathfrak{k}}, \xi_{\mathfrak{k}})$ is a Kuranishi family for $\mathfrak{L}(\mathfrak{k})$.*

Proof. This follows from Proposition 4 and Proposition 3. \square

2.6. The complete picture. Having described the fibres of $\mathbf{Def} \rightarrow \mathbf{Sub}$ in terms of Kuranishi families, we would like to assemble them together and recover the original category \mathbf{Def} . Note that passing to the fibres we lose information about those morphisms in \mathbf{Def} which project to non-identity morphisms in \mathbf{Sub} . That is, these are either (1) homomorphisms of filtered deformations of a single $\mathfrak{k} \subset \mathfrak{g}$, whose associated graded is a nontrivial automorphism of \mathfrak{k} ; or (2) homomorphisms of filtered deformations of different graded subalgebras of \mathfrak{g} .

If the map assigning to each \mathfrak{k} in \mathbf{Sub} the action groupoid $\text{MC}(\mathfrak{L}(\mathfrak{k})) // \exp \mathfrak{L}^0(\mathfrak{k})$ were (pseudo)functorial, we would recover a category fibred over \mathbf{Sub} through the so-called Grothendieck construction. However, \mathbf{Def} is *not* a fibred category over \mathbf{Sub} , and even the map $\mathfrak{k} \mapsto \text{MC}(\mathfrak{L}(\mathfrak{k}))$ isn't functorial – indeed, given a graded Lie algebra monomorphism $\mathfrak{k} \rightarrow \mathfrak{k}'$, we only have a subset of Maurer-Cartan element in $\mathfrak{L}(\mathfrak{k}')$ that project-restrict to $\mathfrak{L}(\mathfrak{k})$. We thus have to perform the construction by hand:

Definition 12. \mathbf{MC} is the category whose objects are pairs (\mathfrak{k}, ϕ) where \mathfrak{k} is an object of \mathbf{Sub} and $\phi \in \text{MC}(\mathfrak{L}(\mathfrak{k}))$. Its morphisms $(\mathfrak{k}, \phi) \rightarrow (\mathfrak{k}', \phi')$ are equivalence classes of triples (g, u, u') where

$$g \in \text{Hom}_{\mathbf{Sub}}(\mathfrak{k}, \mathfrak{k}'), \quad u \in \exp \mathfrak{L}^0(\mathfrak{k}), \quad u' \in \exp \mathfrak{L}^0(\mathfrak{k}')$$

are such that the diagram

$$\begin{array}{ccc} \Lambda^2 \mathfrak{k}'^* & \xrightarrow{u'^{-1} \phi'} & \mathfrak{k}' \\ \downarrow & & \uparrow \\ \Lambda^2 (g\mathfrak{k})^* & \xrightarrow{\text{Ad}_g u \phi} & g\mathfrak{k} \end{array}$$

commutes. Two triples (g, u, u') and (h, v, v') are equivalent if and only if $g = h$ and $u' \circ \text{Ad}_g \circ u = v \circ \text{Ad}_h \circ v'$ as maps $\mathfrak{k} \rightarrow \mathfrak{k}'$.

We have an obvious forgetful functor $\mathbf{MC} \rightarrow \mathbf{Sub}$, and it is easy to see that the fibre $\mathbf{MC}_{\mathfrak{k}}$ is *canonically isomorphic* (not just equivalent) to $\text{MC}(\mathfrak{L}(\mathfrak{k})) // \exp \mathfrak{L}^0(\mathfrak{k})$, and thus equivalent to $\mathbf{Def}_{\mathfrak{k}}$.

Proposition 5. *There is an equivalence of categories between \mathbf{Def} and \mathbf{MC} , compatible with the forgetful functors to \mathbf{Sub} .*

Proof. Just as in the proof of Proposition 4, the construction of Lemma 6 gives a rise to a functor

$$\begin{array}{ccc} \mathbf{MC} & \xrightarrow{F} & \mathbf{Def} \\ & \searrow & \swarrow \\ & \mathbf{Sub} & \end{array}$$

sending (\mathfrak{k}, ϕ) to \mathfrak{k}_{ϕ} together with the tautological map $\iota_{\phi} : \text{gr } \mathfrak{k}_{\phi} \rightarrow \mathfrak{k} \subset \mathfrak{g}$. A pair (g, u) , viewed as a morphism $(\mathfrak{k}, \phi) \rightarrow (\mathfrak{k}', \phi')$ is sent by F to the composite

$$\mathfrak{k}_{\phi} = \mathfrak{k} \xrightarrow{g} \mathfrak{k}' \xrightarrow{u} \mathfrak{k}' = \mathfrak{k}'_{\phi'}.$$

The restriction of F to a fibre is the equivalence $\mathbf{MC}_{\mathfrak{k}} \rightarrow \mathbf{Def}_{\mathfrak{k}}$ of Proposition 4. This in particular shows that F is essentially surjective. Consider now the induced map

$$\text{Hom}_{\mathbf{MC}}((\mathfrak{k}, \phi), (\mathfrak{k}', \phi')) \xrightarrow{F} \text{Hom}_{\mathbf{Def}}((\mathfrak{k}_{\phi}, \iota_{\phi}), (\mathfrak{k}'_{\phi'}, \iota_{\phi'}))$$

on homsets. We shall construct its inverse. Let $U \subset \text{GL}(\mathfrak{k})$, $U' \subset \text{GL}(\mathfrak{k}')$ be the unipotent subgroups identified with $\exp \mathfrak{L}^0(\mathfrak{k})$, $\exp \mathfrak{L}^0(\mathfrak{k}')$ as in the proof of Proposition 4. Recall that the homset on the right hand side consists of pairs (φ, g) where $\varphi : \mathfrak{k}_{\phi} \rightarrow \mathfrak{k}'_{\phi'}$ is a filtered Lie algebra homomorphism and $g \in G_0$ is an element (unique by Lemma 4) such that $\iota_{\phi'} \circ \text{gr } \varphi = \iota_{\phi} \circ \text{Ad}_g$. Since $\iota_{\phi}, \iota_{\phi'}$ are the tautological maps arising from the identification $\mathfrak{k}_{\phi} = \mathfrak{k}$, $\mathfrak{k}'_{\phi'} = \mathfrak{k}'$ as filtered vector spaces, the above condition is simply $\text{gr } \varphi = \text{Ad}_g$. It follows that $\varphi : \mathfrak{k} \rightarrow \mathfrak{k}'$ may be factored as $\varphi = u' \circ \text{Ad}_g \circ u$ for some $u \in U$, $u' \in U'$. Furthermore, the triple (g, u, u') is unique up to equivalence, and thus yields a well-defined element of the homset on the left hand side. It is easy to check that the resulting map is indeed the inverse of F . \square

Although $\mathfrak{k} \mapsto \mathbf{MC}_{\mathfrak{k}}$ does not give a (pseudo)functor from \mathbf{Sub} to groupoids, it becomes functorial once we restrict to isomorphisms. We will only need the following property.

Lemma 8. *An isomorphism $g : \mathfrak{k} \rightarrow \mathfrak{k}'$ in \mathbf{Sub} induces an isomorphism $\mathbf{MC}_g : \mathbf{MC}_{\mathfrak{k}} \rightarrow \mathbf{MC}_{\mathfrak{k}'}$ of groupoids such that for each object (\mathfrak{k}, ϕ) of $\mathbf{MC}_{\mathfrak{k}}$ there is an isomorphism $(\mathfrak{k}, \phi) \rightarrow \mathbf{MC}_g(\mathfrak{k}, \phi)$ in \mathbf{MC} .*

Proof. Given an isomorphism $g : \underline{\mathfrak{X}} \rightarrow \underline{\mathfrak{X}}'$ in **Sub**, we let the functor $\mathrm{MC}_g : \mathbf{MC}_{\underline{\mathfrak{X}}} \rightarrow \mathbf{MC}_{\underline{\mathfrak{X}'}}$ send $(\underline{\mathfrak{X}}, \phi)$ to $(\underline{\mathfrak{X}}', \phi')$ where $\phi'(gX, gY) = g\phi(X, Y)$ for all $X, Y \in \underline{\mathfrak{X}}$. Its action on morphisms sends $u \in \exp \mathcal{L}^0(\underline{\mathfrak{X}})$ to $gug^{-1} \in \exp \mathcal{L}^0(\underline{\mathfrak{X}}')$. \square

Defining a category equivalent to **MC** on the level of Kuranishi families is somewhat cumbersome. Instead, we shall only extend the equivalence $\mathbf{Def}_{\underline{\mathfrak{X}}} \approx M_{\underline{\mathfrak{X}}}$ (discrete) of Corollary 1 to a *full* subcategory of **Def**. That is, we shall also have a way to represent morphisms between objects of $\mathbf{Def}_{\underline{\mathfrak{X}}}$ which map to a *non-trivial* automorphism of $\underline{\mathfrak{X}}$ in **Sub**. Let us first denote by $G_0^{\underline{\mathfrak{X}}} \subset G_0$ the stabiliser of $\underline{\mathfrak{X}}$ under the adjoint action and observe that it acts by automorphisms on $\underline{\mathfrak{X}}$, and thus on $\mathcal{L}(\underline{\mathfrak{X}})$ and $\mathrm{MC}(\mathcal{L}(\underline{\mathfrak{X}}))$. Now, the action of $G_0^{\underline{\mathfrak{X}}}$ is compatible with that of $\exp \mathcal{L}^0(\underline{\mathfrak{X}})$, and thus it descends to the quotient – hence, also to $M_{\underline{\mathfrak{X}}}$ for a Kuranishi family $(M_{\underline{\mathfrak{X}}}, \xi_{\underline{\mathfrak{X}}})$.

Definition 13. Let $\underline{\mathfrak{X}}$ be an object of **Sub**.

- (1) $\mathbf{Def}_{\underline{\mathfrak{X}}}^*$ is the full subcategory of **Def** consisting of objects over $\underline{\mathfrak{X}}$,
- (2) $\mathbf{MC}_{\underline{\mathfrak{X}}}^*$ is the full subcategory of **MC** consisting of objects over $\underline{\mathfrak{X}}$,

Lemma 9. Assume $H^0(\mathcal{L}(\underline{\mathfrak{X}})) = 0$ and let $(M_{\underline{\mathfrak{X}}}, \xi_{\underline{\mathfrak{X}}})$ be a Kuranishi family for $\mathcal{L}(\underline{\mathfrak{X}})$. Then there is an equivalence of categories between $\mathbf{Def}_{\underline{\mathfrak{X}}}^*$ and $M_{\underline{\mathfrak{X}}} // G_0^{\underline{\mathfrak{X}}}$.

Proof. By Proposition 5 way may replace $\mathbf{Def}_{\underline{\mathfrak{X}}}^*$ with $\mathbf{MC}_{\underline{\mathfrak{X}}}^*$. Recalling the equivalence

$$\xi : M_{\underline{\mathfrak{X}}} \rightarrow \mathbf{MC}_{\underline{\mathfrak{X}}} \simeq \mathrm{MC}(\mathcal{L}(\underline{\mathfrak{X}})) // \exp \mathcal{L}^0(\underline{\mathfrak{X}})$$

we need to extend ξ to a commutative diagram of homomorphisms of groupoids

$$\begin{array}{ccc} M_{\underline{\mathfrak{X}}} & \xrightarrow{\xi} & \mathbf{MC}_{\underline{\mathfrak{X}}} \\ \downarrow & & \downarrow \\ M_{\underline{\mathfrak{X}}} // G_0^{\underline{\mathfrak{X}}} & \xrightarrow[\xi^*]{} & \mathbf{MC}_{\underline{\mathfrak{X}}}^*. \end{array}$$

Acting on objects, ξ^* sends $m \in M_{\underline{\mathfrak{X}}}$ to $\xi(m) \in \mathrm{MC}(\mathcal{L}(\underline{\mathfrak{X}}))$. Acting on morphisms, ξ^* sends $g : m \rightarrow m'$ to $[(g, \mathrm{id}, u)] : \xi(m) \rightarrow \xi(m')$ where $u \in \exp \mathcal{L}^0(\underline{\mathfrak{X}})$ is the unique element such that $u * g\xi(m) = \xi(m')$. Since $\mathbf{MC}_{\underline{\mathfrak{X}}}$ has the same set of objects as $\mathbf{MC}_{\underline{\mathfrak{X}}}^*$ and ξ is essentially surjective, so is ξ^* . Furthermore, it is straightforward to check that sending $[(g, u, u')]$ to g gives a well-defined inverse map to

$$\mathrm{Hom}_{M_{\underline{\mathfrak{X}}} // G_0^{\underline{\mathfrak{X}}}}(m, m') \rightarrow \mathrm{Hom}_{\mathbf{MC}_{\underline{\mathfrak{X}}}^*}(\xi(m), \xi(m')), \quad g \mapsto [(g, \mathrm{id}, u)]$$

so that ξ^* is full and faithful. \square

3. ASSOCIATED PARABOLIC GEOMETRIES

3.1. Introduction. The results of the previous section essentially already suggest a classification algorithm. Using $\pi_0(-)$ to denote the set of isomorphism classes of objects of a groupoid, we have by Lemma 5 Proposition 5 that $\pi_0(\mathbf{Germ})$ forms a subset of the set of isomorphism classes of objects of the small category **MC**. By Lemma 9 we may already identify $\pi_0(\mathbf{MC}_{\underline{\mathfrak{X}}}^*) \simeq M_{\underline{\mathfrak{X}}} // G_0^{\underline{\mathfrak{X}}}$ for each $\underline{\mathfrak{X}}$ in **Sub** satisfying $H^0(\mathcal{L}(\underline{\mathfrak{X}})) = 0$. It will turn out that all subalgebras relevant for our classification

problem do satisfy this condition. Now, for two objects of \mathbf{MC} to be isomorphic, the underlying graded subalgebras of \mathfrak{g} must be conjugate by an element of G_0 . Hence, choosing for each G_0 -conjugacy class a representative, we may embed the relevant subset of $\pi_0(\mathbf{Germ})$ into the disjoint union of $M_{\mathfrak{k}}/G_0^{\mathfrak{k}}$ as \mathfrak{k} runs through the chosen representatives. The Kuranishi families and their quotients may be efficiently computed in a completely algorithmic way. The remaining problem is to identify those points in each $M_{\mathfrak{k}}/G_0^{\mathfrak{k}}$ that correspond to a class of objects of \mathbf{Germ} .

Recall that the embedding $\mathbf{Germ} \rightarrow \mathbf{GermSym} \approx \mathbf{MC}$ admits a left adjoint, giving rise to an endofunctor (idempotent monad) on $\mathbf{GermSym}$ sending $(\mathcal{D}, \mathfrak{k})$ to $(\mathcal{D}, \text{sym } \mathcal{D})$. That is, the objects of \mathbf{Germ} are precisely those satisfying $\mathfrak{k} = \text{sym } \mathcal{D}$ (or just $\dim \mathfrak{k} = \dim \text{sym } \mathcal{D}$), need to recognise them is to be able to compute the symmetry algebra $\text{sym } \mathcal{D}$ (or just its dimension). That can be achieved using the methods of parabolic geometry.

3.2. Algebraic Cartan connections.

Definition 14. Let \mathfrak{k} be a Lie algebra, and $\mathfrak{l} \subset \mathfrak{k}$ a subalgebra. An algebraic Cartan connection of type $(\mathfrak{g}, \mathfrak{p})$ on $(\mathfrak{k}, \mathfrak{l})$ is a linear map $\bar{\omega} : \mathfrak{k} \rightarrow \mathfrak{g}$ such that

- (1) $\bar{\omega}(\mathfrak{l}) \subset \mathfrak{p}$, inducing
- (2) $\mathfrak{k}/\mathfrak{l} \rightarrow \mathfrak{g}/\mathfrak{p}$ an isomorphism,
- (3) $\bar{\omega}([X, Y]) = [\bar{\omega}X, \bar{\omega}Y]$ whenever $X \in \mathfrak{l}$.

Definition 15. Let \mathfrak{k} be a filtered Lie algebra. An algebraic Cartan connection $\bar{\omega} : \mathfrak{k} \rightarrow \mathfrak{g}$ on $(\mathfrak{k}, \mathfrak{k}^0)$ is *regular* if it is filtration preserving, and if $\text{gr } \bar{\omega} : \text{gr } \mathfrak{k} \rightarrow \mathfrak{g}$ is a graded Lie algebra homomorphism.

Definition 16. \mathbf{Cartan}^a is the category whose objects are pairs $(\mathfrak{k}, \bar{\omega})$ where \mathfrak{k} is a filtered Lie algebra, and $\bar{\omega} : \mathfrak{k} \rightarrow \mathfrak{g}$ an *injective* regular algebraic Cartan connection of type $(\mathfrak{g}, \mathfrak{p})$ on $(\mathfrak{k}, \mathfrak{k}^0)$. Its morphisms from $(\mathfrak{k}, \bar{\omega})$ to $(\mathfrak{k}', \bar{\omega}')$ are pairs (φ, p) where $\varphi : \mathfrak{k} \rightarrow \mathfrak{k}'$ is a filtered Lie algebra homomorphism and $p \in P$ is such that $\text{Ad}_p \circ \bar{\omega} = \bar{\omega}' \circ \varphi$.

Observe that the assignment sending $(\mathfrak{k}, \bar{\omega})$ to $(\mathfrak{k}, \text{gr } \bar{\omega})$ extends to a forgetful functor

$$\mathbf{Cartan}^a \rightarrow \mathbf{Def}.$$

Given an algebraic Cartan connection $\bar{\omega} \in \mathfrak{k}^* \otimes \mathfrak{g}$ on $(\mathfrak{k}, \mathfrak{l})$, we define its curvature $\bar{\Omega} \in \Lambda^2 \mathfrak{k}^* \otimes \mathfrak{g}$ by the formula

$$\bar{\Omega} = d^{\mathfrak{k}} \bar{\omega} + \frac{1}{2} [\bar{\omega} \wedge \bar{\omega}]_{\mathfrak{g}}$$

where $d^{\mathfrak{k}}$ denotes the differential in the complex $C^{\bullet}(\mathfrak{k}) \otimes \mathfrak{g}$ (i.e. with \mathfrak{k} acting trivially on \mathfrak{g}), and $[\cdot, \cdot]_{\mathfrak{g}} : \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$ is the bracket in \mathfrak{g} . By the property (3) in Definition 14, $\bar{\Omega} : \Lambda^2 \mathfrak{k} \rightarrow \mathfrak{g}$ factors through $\Lambda^2(\mathfrak{k}/\mathfrak{l})$. Then, by property (2) we may form the ‘curvature function’ $\bar{\kappa} \in \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ defined unambiguously by

$$\bar{\Omega}(X, Y) = \bar{\kappa}(\bar{\omega}(X), \bar{\omega}(Y)) \quad \text{for all } X, Y \in \mathfrak{k}.$$

As is customary, we identify $(\mathfrak{g}/\mathfrak{p})^*$ with \mathfrak{p}_+ as representations of P , and view $\bar{\kappa}$ as a 2-chain in the complex $C_{\bullet}(\mathfrak{p}_+, \mathfrak{g})$ computing Lie algebra homology of \mathfrak{p}_+ with values in \mathfrak{g} . If \mathfrak{k} is filtered, $\mathfrak{l} = \mathfrak{k}^0$ and $\bar{\omega}$ is regular, it follows that $\bar{\kappa} \in C_{\bullet}(\mathfrak{p}_+, \mathfrak{g})^1$. We say that $\bar{\omega}$ is *normal* if $\bar{\kappa}$ is a 2-cycle.

Lemma 10. $\bar{\kappa} \in C_2(\mathfrak{p}_+, \mathfrak{g})$ is annihilated by the adjoint action of $\bar{\omega}(\mathfrak{l}) \subset \mathfrak{p}$.

Proof. This is an algebraic version of the Bianchi identity, using the definition of $\bar{\kappa}$ and $\bar{\Omega}$ and property (3) of Definition 14. \square

The key result connecting algebraic Cartan connections to filtered deformations of graded subalgebras of \mathfrak{g} is the following Proposition. Its proof will be given in the next subsection, after a further excursion into Cartan geometries on homogeneous spaces.

Proposition 6. *Let \mathbf{Cartan}_n^a denote the full subcategory of \mathbf{Cartan}^a consisting of objects $(\mathfrak{k}, \bar{\omega})$ where $\bar{\omega}$ is normal. Then the restriction of the forgetful functor induces an equivalence of categories between \mathbf{Cartan}_n^a and \mathbf{Def} .*

3.3. Geometries on a homogeneous space. Recall the notation of subsection 1.5. We introduce a number of notions describing invariant Cartan geometries on a fixed homogeneous space. The qualification ‘of type (\mathfrak{g}, P) ’ is implicitly understood.

Definition 17. Let K be a Lie group, and L a closed subgroup.

- (1) An *equivariant P -principal bundle* on K/L is a right P -principal bundle $\mathcal{G} \rightarrow K/L$ together with a left K -action on \mathcal{G} commuting with the right P -action.
- (2) A type Cartan connection $\omega \in \Omega_{\mathcal{G}}^1 \otimes \mathfrak{g}$ on an equivariant P -principal bundle $\mathcal{G} \rightarrow K/L$ is *invariant* if $L_k^* \omega = \omega$ for all $k \in K$.
- (3) An *invariant Cartan geometry* on K/L is an equivariant P -principal bundle $\mathcal{G} \rightarrow K/L$ together with an invariant Cartan connection $\omega \in \Omega_{\mathcal{G}}^1 \otimes \mathfrak{g}$.

Given a pair of Cartan geometries (\mathcal{G}, ω) and (\mathcal{G}', ω') of type (\mathfrak{g}, P) on M , a gauge transformation between the two is a bundle morphism $f : \mathcal{G} \rightarrow \mathcal{G}'$ such that $f^* \omega' = \omega$. Note that for M connected, f is uniquely determined by the image of a single point of \mathcal{G} . In particular, the only gauge transformation from (\mathcal{G}, ω) to itself is the identity.

Given Cartan geometries $(\mathcal{G} \rightarrow M, \omega)$, $(\mathcal{G}' \rightarrow M', \omega')$ of type (\mathfrak{g}, P) and a local diffeomorphism $h : M \rightarrow M'$, one forms the pullback Cartan geometry $(h^* \mathcal{G}, \tilde{h}^* \omega)$ where $\tilde{h} : h^* \mathcal{G}' \rightarrow \mathcal{G}$ is the natural projection, and considers gauge transformations $f : \mathcal{G} \rightarrow h^* \mathcal{G}'$. These are, equivalently, local diffeomorphisms $\bar{f} : \mathcal{G} \rightarrow \mathcal{G}'$ lifting h and such that $\bar{f}^* \omega' = \omega$. Furthermore, given just a germ h at $m \in M$ of a local diffeomorphism $M \rightarrow M'$, it makes sense to consider germs of a local gauge transformations from (\mathcal{G}, ω) to $h^*(\mathcal{G}', \omega')$. These may be identified with equivalence classes of pairs (U, f_U) where $U \subset M$ is an open neighbourhood of m , and $f_U : \mathcal{G}|_U \rightarrow \mathcal{G}'$ is a local diffeomorphism such that $f_U^* \omega' = \omega$ and $f|_U$ descends to a local diffeomorphism $U \rightarrow M'$ representing h .

Definition 18. \mathbf{Cartan}^g is the category whose objects are $(K, L, \mathcal{G}, \omega)$ where K is a Lie group, L a connected closed subgroup and (\mathcal{G}, ω) an invariant regular Cartan geometry on K/L such that L acts freely in the fibre of \mathcal{G} over the origin. Its morphisms from $(K, L, \mathcal{G}, \omega)$ to $(K', L', \mathcal{G}', \omega')$ are pairs (φ, f) where $\varphi : \mathfrak{k} \rightarrow \mathfrak{k}'$ is a Lie algebra monomorphism, $\varphi(\mathfrak{l}) \subset \mathfrak{l}'$ and f is a germ at the origin of a local gauge transformation from (\mathcal{G}, ω) to $h_\varphi^*(\mathcal{G}', \omega')$, with h_φ being the germ at the origin of a local diffeomorphism $K/L \rightarrow K'/L'$ induced by φ .

Lemma 11. *Let $(K, L, \mathcal{G}, \omega)$ be an object of \mathbf{Cartan}^g together with a point $e \in \mathcal{G}$ over the origin. Then the orbit map*

$$\lambda_e : K \rightarrow \mathcal{G}, \quad k \mapsto ke$$

induces a Lie group monomorphism $L \rightarrow P$, and $\lambda_e^* \omega$ viewed as a left-invariant \mathfrak{g} -valued one-form on K is an algebraic Cartan connection on \mathfrak{k} .

Proof. The homomorphism $L \rightarrow P$ maps $\ell \in L$ to $p \in P$ such that $\ell e = ep^{-1}$. It is injective by freeness of the action of L on the fibre eP . Furthermore, letting $\bar{\omega} = \lambda_e^* \omega$ we have (1) $\bar{\omega}(\mathfrak{l}) \subset \mathfrak{p}$ since L preserves the origin; (2) $\mathfrak{k}/\mathfrak{l} \rightarrow \mathfrak{g}/\mathfrak{p}$ is an isomorphism since ω is a Cartan connection over K/L ; (3) $[\bar{\omega}X, \bar{\omega}Y] - \bar{\omega}([X, Y]) = 0$ for $X \in \mathfrak{l}$ by horizontality of the Cartan curvare of ω . \square

Lemma 12. *Let $(\mathfrak{k}, \bar{\omega})$ be an object of \mathbf{Cartan}^a . Then there exists a Lie group K with Lie algebra \mathfrak{k} , and a Lie group monomorphism $K \supset K^0 \rightarrow P$ lifting $\bar{\omega} : \mathfrak{k}^0 \rightarrow \mathfrak{p}$.*

Proof. By the universal property of Tanaka prolongation, the map $\mathfrak{g} \rightarrow \text{Hom}(\mathfrak{g}_-, \mathfrak{g})$ is injective, and so is thus the adjoint representation $\text{gr } \mathfrak{k} \rightarrow \text{End } \text{gr } \mathfrak{k}$. Since the adjoint representation of $\text{gr } \mathfrak{k}$ is the associated graded map of the adjoint representation of \mathfrak{k} , it follows that the latter is faithful. We may then let K be the connected subgroup of $\text{GL}(\mathfrak{k})$ with Lie algebra $\mathfrak{k} \subset \text{End } \mathfrak{k}$. Now, let $P_{\mathfrak{k}} \subset P$ be the stabiliser of $\bar{\omega}(\mathfrak{k}) \subset \mathfrak{g}$ for the adjoint action. Since $\text{gr } \bar{\omega}(\text{gr } \mathfrak{k}) \supset \mathfrak{g}_-$, it follows that the natural Lie group homomorphism $P_{\mathfrak{k}} \rightarrow \text{GL}(\mathfrak{k})$ induced by $\bar{\omega}$ is injective. In particular, we have a Lie algebra monomorphism $\mathfrak{p}_{\mathfrak{k}} \rightarrow \text{End } \mathfrak{k}$ whose image contains \mathfrak{k}^0 ; the resulting homomorphism $\mathfrak{k}^0 \rightarrow \mathfrak{p}$ coincides with the restriction of $\bar{\omega}$. Since K^0 is connected, it follows that $K^0 \subset P_{\mathfrak{k}}$ as subgroups of $\text{GL}(\mathfrak{k})$, giving rise to the desired Lie group homomorphism $K^0 \rightarrow P$ lifting $\bar{\omega}$. \square

The essence of the following result is contained in [6, Prop. 1.5.15]. It has an important interpretation in terms of the practical implementation of homogeneous parabolic geometries in **DifferentialGeometry**; I am immensely indebted to Ian Anderson for our discussions on these matters.

Lemma 13. *There is an equivalence of categories between \mathbf{Cartan}^a and \mathbf{Cartan}^g .*

Proof. To construct a functor $F : \mathbf{Cartan}^g \rightarrow \mathbf{Cartan}^a$, let us fix for each object $(K, L, \mathcal{G}, \omega)$ a point $e \in \mathcal{G}$ in the fibre over the origin. By Lemma 11 we then have an algebraic Cartan connection $\lambda_e^* \omega$ on \mathfrak{k} . Defining a filtration \mathfrak{k}^\bullet on \mathfrak{k} by $\mathfrak{k}^i = (\lambda_e^* \omega)^{-1}(\mathfrak{g}^i)$ we set $F(K, L, \mathcal{G}, \omega) = (\mathfrak{k}^\bullet, \lambda_e^* \omega)$. Given a morphism $(\varphi, f) : (K, L, \mathcal{G}, \omega) \rightarrow (K', L', \mathcal{G}', \omega')$ we let $F(\varphi, f) = (\varphi, p)$ with $p \in P$ such that $f(e) = e' \cdot p$ (by abuse of notation we identify e' with the unique element in the fibre of $h_\varphi^* \mathcal{G}'$ over the origin mapping to $e' \in \mathcal{G}'$). Letting \tilde{h}_φ be the germ at e_K of a local embedding $K \rightarrow K'$ induced by φ , we then have

$$R_p^{-1} \circ f \circ \lambda_e = \lambda_{e'} \circ \tilde{h}_\varphi$$

as germs at e_K of maps $K \rightarrow \mathcal{G}'$, so that

$$\text{Ad}_p \circ \lambda_e^* \omega = \lambda_{e'}^* \omega' \circ \varphi$$

and we have indeed defined a morphism in \mathbf{Cartan}^a . Finally, compatibility with composition is a bit tedious, but not difficult to check.

To construct a functor $G : \mathbf{Cartan}^a \rightarrow \mathbf{Cartan}^g$, consider for each object $(\mathfrak{k}, \bar{\omega})$ the Lie groups $K \supset K^0$ as in Lemma 12. Set $G(\mathfrak{k}, \bar{\omega}) = (K, K^0, K \times^{K^0} P, \omega)$ where ω is the unique invariant Cartan connection such that $\lambda_{(e_K, e_P)}^* \omega = \bar{\omega}$. Furthermore, given a morphism $(\varphi, p) : (\mathfrak{k}, \bar{\omega}) \rightarrow (\mathfrak{k}', \bar{\omega}')$ let $G(\varphi, p) = (\varphi, R_p)$ where we identify $h_\varphi^*(K' \times^{K'^0} P)$ with the germ at the origin of $K \times^{K^0} P$ via \tilde{h}_φ . Again, it is straightforward to check that we have defined a functor.

There is a natural isomorphism $\text{id} \rightarrow FG$ of endofunctors on \mathbf{Cartan}^a , whose component at $(\mathfrak{k}, \bar{\omega})$ is $(\text{id}_{\mathfrak{k}}, p)$ where $p \in P$ is such that $(e_K, p) \in K \times^{K^0} P$ is the point used in the construction of F . In the opposite direction, we have the natural isomorphism $\text{id} \rightarrow GF$ of endofunctors on \mathbf{Cartan}^g , whose component at $(K, L, \mathcal{G}, \omega)$ is $(\text{id}_{\mathfrak{k}}, f)$ where f is a germ of a gauge transformation sending the point $e \in \mathcal{G}$ used in the construction of F to $(e_K, e_P) \in K \times^L P$ (we identify a neighbourhood of identity in K with a neighbourhood of identity in the group used in the construction of G , and embed L in P as in Lemma 11). \square

Lemma 14. *Let $(K, L, \mathcal{G}, \omega)$ be an object of \mathbf{Cartan}^g and $(\mathfrak{k}, \bar{\omega})$ an object of \mathbf{Cartan}^a such that the two become isomorphic under the equivalence of categories $\mathbf{Cartan}^g \approx \mathbf{Cartan}^a$ of Lemma 13. Then the curvature function $\kappa : \mathcal{G} \rightarrow C_2(\mathfrak{p}_+, \mathfrak{g})$ factors through the P -orbit of $\bar{\kappa}$.*

Proof. This follows from the identity $\lambda_e^* d\omega = d^{\mathfrak{k}} \lambda_e^* \omega$ where we view λ_e^* as pulling back K -invariant \mathfrak{g} -valued one-forms on \mathcal{G} to elements of $\Lambda^{\bullet} \mathfrak{k}^* \otimes \mathfrak{g}$. Indeed, letting $\bar{\omega} = \lambda_e^* \omega$ we then have $\bar{\Omega} = \lambda_e^* \Omega$ where $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ and $\bar{\kappa} = \lambda_e^* \kappa$ where κ is the K -invariant curvature function on \mathcal{G} . In general it follows that κ factors through the P -orbit of $\bar{\kappa}$, and furthermore the latter statement is stable under isomorphisms in \mathbf{Cartan}^a . \square

By Lemma 14, the equivalence of Lemma 13 restricts to an equivalence between \mathbf{Cartan}_n^a and \mathbf{Cartan}_n^g , the full subcategory of \mathbf{Cartan}^g consisting of those $(K, L, \mathcal{G}, \omega)$ for which ω is normal.

Proof of Proposition 6. We need to construct an essential inverse to the obvious forgetful functor $\mathbf{Cartan}_n^a \rightarrow \mathbf{Def}$. This is equivalent to providing an essential inverse to the functor $A : \mathbf{Cartan}_n^g \rightarrow \mathbf{Def}$, a composite of the above with $F : \mathbf{Cartan}_n^g \rightarrow \mathbf{Cartan}_n^a$ as in the proof of Lemma 13. Recall that we choose for each $(K, L, \mathcal{G}, \omega)$ a point $e \in \mathcal{G}$ over the origin; then $A(K, L, \mathcal{G}, \omega) = (\mathfrak{k}, \text{gr } \lambda_e^* \omega)$ where the filtration on \mathfrak{k} is induced by $\lambda_e^* \omega$. On morphisms we have $A(\varphi, f) = (\varphi, g)$ where g is the class in G_0 of the element $p \in P$ such that $f(e) = e'p$.

To define a functor in the opposite direction, choose for each (\mathfrak{k}, ι) in \mathbf{Def} Lie groups $K^0 \subset K$ with Lie algebras $\mathfrak{k}^0 \subset \mathfrak{k}$ such that K acts faithfully on K/K^0 , and let $\mathcal{D}_{\mathfrak{k}}$ be the invariant distribution on K corresponding to $\mathfrak{k}^{-1}/\mathfrak{k}^0$. Use Proposition 1 to construct a regular normal Cartan geometry $(\mathcal{G}_{\mathfrak{k}}, \omega_{\mathfrak{k}})$ over K/L for the underlying datum of $\mathcal{D}_{\mathfrak{k}}$. Set $B(\mathfrak{k}, \iota) = (K, K^0, \mathcal{G}_{\mathfrak{k}}, \omega_{\mathfrak{k}})$. Again by Proposition 1, given a morphism $(\varphi, g) : (\mathfrak{k}, \iota) \rightarrow (\mathfrak{k}', \iota')$ in \mathbf{Def} , there is a unique germ of a gauge transformation $f : (\mathcal{G}_{\mathfrak{k}}, \omega_{\mathfrak{k}}) \rightarrow h_{\varphi}^*(\mathcal{G}_{\mathfrak{k}'}, \omega_{\mathfrak{k}'})$. Set $B(\varphi, g) = (\varphi, f)$. We have thus defined a functor $B : \mathbf{Def} \rightarrow \mathbf{Cartan}_n^g$.

Now, $AB \simeq \text{id}$ as endofunctors on \mathbf{Cartan}_n^g by Proposition 1 (uniqueness of a regular normal parabolic geometry up to unique gauge). On the other hand $BA \simeq \text{id}$ as endofunctors on \mathbf{Def} since any two graded Lie algebra monomorphisms $\text{gr } \mathfrak{k} \rightrightarrows \mathfrak{g}$ whose image contains \mathfrak{g}_- differ by a unique element of G_0 . \square

3.4. Invariants and symmetries. Observe that Proposition 6 implies a slightly stronger statement: for every object (\mathfrak{k}, ι) of \mathbf{Def} , we may lift $\iota : \text{gr } \mathfrak{k} \rightarrow \mathfrak{g}$ to a regular normal algebraic Cartan connection $\bar{\omega} : \mathfrak{k} \rightarrow \mathfrak{g}$, and furthermore that latter lift is unique up to the adjoint action of P_+ on \mathfrak{g} . Now, by Lemma 14 the curvature function $\bar{\kappa} \in Z_2(\mathfrak{p}_+, \mathfrak{g})$ coincides with the curvature function of the parabolic geometry associated with the germ of a distribution described by \mathfrak{k} , and

in particular its homology class $\bar{\kappa}_H \in H_2(\mathfrak{p}_+, \mathfrak{g})$ is the harmonic curvature. More precisely, given (\mathfrak{k}, ι) the curvature $\bar{\kappa}$ is defined uniquely up to P_+ -conjugacy, and the harmonic curvature $\bar{\kappa}_H$ is an honest invariant.

Lemma 15. *The assignment sending an object (\mathfrak{k}, ι) of **Def** to the harmonic curvature of a regular normal algebraic Cartan connection lifting ι extends to a functor $\mathbf{Def} \rightarrow H_2(\mathfrak{p}_+, \mathfrak{g})^1 // G_0$.*

Proof. The functor in question sends a morphism $(\varphi, g) : (\mathfrak{k}, \iota) \rightarrow (\mathfrak{k}', \iota')$ to $g \in G_0$. \square

We will thus be able to compute the basic invariant $\bar{\kappa}_H$ of (\mathfrak{k}, ι) by computing a regular, normal algebraic Cartan connection lifting ι , and then its harmonic curvature. We shall not go into the technical details of this calculation in this paper. As we have stated in the introductory subsection, the main reason for using the parabolic geometric description is an explicit algorithm computing the infinitesimal symmetries. Recall from [6, Lemma 1.5.12] that the infinitesimal symmetries of a Cartan geometry $(\mathcal{G} \rightarrow M, \omega)$ of type (\mathfrak{g}, P) , viewed as vector fields on \mathcal{G} whose flow preserves ω , are in one-to-one correspondence with sections of the associated bundle $\mathcal{G} \times^P \mathfrak{g}$ parallel for the *modified tractor connection*. Identifying $\Gamma(M, \mathcal{G} \times^P \mathfrak{g})$ with $C^\infty(\mathcal{G}, \mathfrak{g})^P$, we say that a P -equivariant function $x : \mathcal{G} \rightarrow \mathfrak{g}$ is parallel if

$$dx + \omega \cdot x + \tilde{\kappa} \cdot x = 0$$

where $\tilde{\kappa} \in \Omega_{\mathcal{G}}^1 \otimes \text{End } \mathfrak{g}$ is given by $\tilde{\kappa}(\xi) \cdot X = \kappa(\omega(\xi), X)$ for $\xi \in T\mathcal{G}$ and $X \in \mathfrak{g}$. Translating this prescription to the algebraic setting, we have the following.

Lemma 16. *Let $(\mathfrak{k}, \bar{\omega})$ be an object of \mathbf{Cartan}_n^a . Define*

$$\alpha : \mathfrak{k} \rightarrow \text{End } \mathfrak{g}, \quad \alpha(X)Y = [\bar{\omega}X, Y] + \bar{\kappa}(\bar{\omega}X, Y) \quad \text{for all } X \in \mathfrak{k}, Y \in \mathfrak{g}$$

and let

$$R = d^{\mathfrak{k}}\alpha + \frac{1}{2}[\alpha \wedge \alpha]_{\text{End } \mathfrak{g}}$$

where $d^{\mathfrak{k}}$ is the differential in $C^\bullet(\mathfrak{k}) \otimes \text{End } \mathfrak{g}$ (trivial action on coefficients) and $[\cdot, \cdot]_{\text{End } \mathfrak{g}}$ the bracket in $\text{End } \mathfrak{g}$. Let $\mathfrak{s} \subset \mathfrak{g}$ be the subspace annihilated by endomorphisms of the form

$$\alpha(X_1) \cdots \alpha(X_{r-2})R(X_{r-1}, X_r), \quad r \geq 0, \quad X_1, \dots, X_r \in \mathfrak{k}.$$

Then $\dim \mathfrak{s} = \dim \text{sym } \mathcal{D}$ where \mathcal{D} is a germ of a Monge distribution corresponding to $(\mathfrak{k}, \bar{\omega})$ under the composite forgetful functor $\mathbf{Cartan}_n^a \rightarrow \mathbf{Germ}$.

For the sake of brevity, the following is a sketch of an argument. I owe the idea of using formal holonomy to Ian Anderson.

Proof. Consider a regular normal invariant Cartan geometry $(K, K^0, \mathcal{G}, \omega)$ corresponding to $(\mathfrak{k}, \bar{\omega})$ under the equivalence $\mathbf{Cartan}_n^a \approx \mathbf{Cartan}_n^g$, together with a point $e \in \mathcal{G}$ over the origin such that $\bar{\omega} = \lambda_e^* \omega$. Then α and R correspond precisely to the ‘modified tractor connection’ on $\mathcal{G} \times^P \mathfrak{g}$ and its curvature, pulled back by λ_e . Since every parallel section, represented by a map $x : \mathcal{G} \rightarrow \mathfrak{g}$, is determined by its value $x(e) \in \mathfrak{g}$, it is enough to find the image in \mathfrak{g} of the space of germs of parallel sections under the evaluation map at e . Since the latter map is an injection into a finite-dimensional space, we may in fact consider infinite jets, rather than germs, of parallel sections. Then the desired subspace of \mathfrak{g} is the kernel of the ‘formal holonomy group’, i.e. the subspace $\mathfrak{s} \subset \mathfrak{g}$ defined in the Lemma. \square

4. CLASSIFICATION ALGORITHM

4.1. Introduction. We have established a sufficient foundation for setting up our classification algorithm. Recall that we have passed through the following chain of categories

$$\mathbf{Germ} \rightleftharpoons \mathbf{GermSym} \approx \mathbf{Def} \approx \mathbf{Cartan}_n^a, \quad \mathbf{Def}_{\underline{\mathfrak{k}}}^* \approx M_{\underline{\mathfrak{k}}} // G_0^{\underline{\mathfrak{k}}}$$

where the latter holds if $H^0(\mathcal{L}(\underline{\mathfrak{k}})) = 0$. In addition, given an object $(\mathfrak{k}, \bar{\omega})$ of \mathbf{Cartan}_n^a we may compute the harmonic curvature $\bar{\kappa}_H$ and the dimension of the symmetry algebra of the corresponding distribution in terms of $\bar{\omega}$ (see Lemmas 15 and 16).

The algorithm will pass through equivalence classes of graded subalgebras in \mathfrak{g} . Since we are interested in *non-flat* 2-transitive models with vanishing scalar component of harmonic curvature, we may from the outset restrict to conjugacy classes represented by $\underline{\mathfrak{k}}$ such that $\dim \underline{\mathfrak{k}}_0 \geq 2$ and $\underline{\mathfrak{k}}_0$ annihilates a non-zero element of the quintic component of $H_2(\mathfrak{p}_+, \mathfrak{g})$. Indeed, by Lemma 10, the harmonic curvature κ_H of any (\mathfrak{k}, ι) in $\mathbf{Def}_{\underline{\mathfrak{k}}}$ is invariant under $\bar{\omega}(\mathfrak{k}^0) \subset \mathfrak{p}$, and thus under $\text{gr } \bar{\omega}(\mathfrak{k}_0) = \underline{\mathfrak{k}}_0 \subset \mathfrak{g}_0$, where $\bar{\omega}$ is some regular, normal algebraic Cartan connection lifting ι . These graded subalgebras turn out to be very simple:

Lemma 17. *Suppose $\underline{\mathfrak{k}} \subset \mathfrak{g}$ is a graded subalgebra containing \mathfrak{g}_- and such that $\underline{\mathfrak{k}}_0$ annihilates a nonzero element of the quintic component of $H_2(\mathfrak{p}_+, \mathfrak{g})^1$. Then $\underline{\mathfrak{k}}_i = 0$ for all $i > 0$.*

Proof. Let $W \subset H_2(\mathfrak{p}_+, \mathfrak{g})^1$ denote the quintic component; it decomposes into one-dimensional weight subspaces with weights $\lambda_j = \alpha_2 + (j-2)\alpha_1$, $0 \leq j \leq 5$. We want to show, for each $i > 0$, that the map

$$\mathfrak{g}_i \otimes W \rightarrow \mathfrak{g}_i \otimes W, \quad X \otimes v \mapsto [X, -] \cdot v$$

is an isomorphism (where we identify $\mathfrak{g}_i \simeq \mathfrak{g}_{-i}^*$ via the Killing form). Now, the above is G_0 -equivariant and one easily checks that the multiplicity of each irreducible component of $\mathfrak{g}_i \otimes W$ is 1, whence the map is a scalar on every irreducible component. In particular, it is enough to check that $[X_\alpha, -] \cdot v_j \neq 0$ for every weight vector $v_j \in W_{\lambda_j}$ and root vector $X_\alpha \in \mathfrak{g}_\alpha$, $\text{ht } \alpha = i$. Using the Cartan matrix of \mathfrak{g} one computes

$$[X_\alpha, X_{-\alpha}] \cdot v_j = \langle \lambda_j, \alpha^\vee \rangle v_j \neq 0$$

whenever either $j \in \{0, 5\}$ or $\alpha = \alpha_3$. In the remaining cases, we have

$$[X_\alpha, X_{-\alpha \pm \alpha_1}] \cdot v_j \sim X_{\pm \alpha_1} \cdot v_j = v_{j \pm 1} \neq 0$$

whenever $-\alpha \pm \alpha_1$ is a root (which holds at least for one choice of sign). \square

In particular, we find that all graded subalgebras $\underline{\mathfrak{k}}$ we need to consider do satisfy the cohomology vanishing property, and thus their filtered deformations admit universal global Kuranishi families.

Corollary 2. *Suppose $\underline{\mathfrak{k}} \subset \mathfrak{g}$ is a graded subalgebra containing \mathfrak{g}_- and such that $\underline{\mathfrak{k}}_0$ annihilates a nonzero element of the quintic component of $H_2(\mathfrak{p}_+, \mathfrak{g})^1$. Then $H^{1,1}(\underline{\mathfrak{k}}, \underline{\mathfrak{k}}) = H^0(\mathcal{L}(\underline{\mathfrak{k}})) = 0$.*

Proof. First, by Lemma 17, we have $\underline{\mathfrak{k}} = \mathfrak{g}_- \oplus \underline{\mathfrak{k}}_0$. Now, recall that $H^{1,1}(\underline{\mathfrak{k}}, \underline{\mathfrak{k}})$ is the space of positive degree derivations of $\underline{\mathfrak{k}}$. Note that since $\underline{\mathfrak{k}}_0$ embeds into $\text{End } \mathfrak{g}_{-1}$, every $\delta \in H_i^1(\underline{\mathfrak{k}}, \underline{\mathfrak{k}})$, $i > 0$ is determined by the induced map $\bigotimes^{i+1} \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$

sending $X_0 \otimes \cdots \otimes X_i$ to $[X_i, [\cdots [X_1, \delta X_0]]]$. Thus, by the universal property of Tanaka prolongation, we have an embedding $\mathfrak{k} \oplus H^{1,1}(\mathfrak{k}, \mathfrak{k}) \rightarrow \mathfrak{g}$ as a graded Lie subalgebra. Using Lemma 17 again, we find that the latter subalgebra is contained in non-positive degrees, whence $H^{1,1}(\mathfrak{k}, \mathfrak{k}) = 0$. \square

The representing map $\xi_{\mathfrak{k}} : M_{\mathfrak{k}} \rightarrow \text{MC } \mathfrak{L}(\mathfrak{k})$ defines for each point $m \in M_{\mathfrak{k}}$ a filtered deformation \mathfrak{k}_m of \mathfrak{k} together with a monomorphism $\iota_m : \text{gr } \mathfrak{k}_m \rightarrow \mathfrak{g}$, and we may furthermore choose a lift of ι_m to a regular, normal algebraic Cartan connection $\bar{\omega}_m : \mathfrak{k}_m \rightarrow \mathfrak{g}$. This allows us to compute the harmonic curvature $\bar{\kappa}_m$ and the symmetry dimension $d(m)$ computed as in Lemma 16. Points m at which $d(m) > \dim \mathfrak{k}$ are rejected. In practice, these computations are performed globally over $M_{\mathfrak{k}}$: viewing the latter as an algebraic subvariety in $H^{2,1}(\mathfrak{k}, \mathfrak{k})$, the family $\{\mathfrak{k}_m\}$ and the data of $\bar{\omega}_m$ are seen as a filtered Lie algebra and an algebraic Cartan connection over the ring $\mathbb{R}[M_{\mathfrak{k}}]$.

It is a non-trivial observation that the method computing a regular normal algebraic Cartan connection may be carried out over a ring and gives rise to a well-defined connection upon reduction to the residue field at each point. The situation with the symmetry dimension $d(m)$ is different, as $d : M_{\mathfrak{k}} \rightarrow \mathbb{Z}$ is Zariski upper-semicontinuous. In practice, one computes the symmetry dimension at the generic point of each irreducible component of $M_{\mathfrak{k}} \otimes_{\mathbb{R}} \mathbb{C}$, applying the prescription of Lemma 16 in terms of linear algebra over the field of rational functions. If it is strictly greater than $\dim \mathfrak{k}$, the component may be rejected (by upper semi-continuity). Otherwise, one would proceed to find a proper Zariski-closed subset at which d jumps, decompose it into irreducible components, compute d at generic points etc. In the end, we shall not go that far in the present article: the families we describe might contain finitely many special points corresponding to models that have already been included in another family. Here by $M_{\mathfrak{k}} \otimes_{\mathbb{R}} \mathbb{C}$ we mean the subvariety of $\mathbb{C} \otimes_{\mathbb{R}} H^1(\mathfrak{L}(\mathfrak{k}))$ cut out by the quadratic condition $[\Phi^{-1}x, \Phi^{-1}x] \in B^2(\mathfrak{L}(\mathfrak{k}))$ as in the proof of Proposition 3; in particular, the algebraic set $M_{\mathfrak{k}} \subset H^1(\mathfrak{L}(\mathfrak{k}))$ is its set of real points. Interestingly, the irreducible components of $M_{\mathfrak{k}} \otimes_{\mathbb{R}} \mathbb{C}$ turn out to be defined over \mathbb{R} , and in fact rational.

4.2. Ingredients and recipe. We review the computational tasks forming the basic ingredients of our method:

- (1) to parameterise G_0 -conjugacy classes of subalgebras $\mathfrak{k}_0 \subset \mathfrak{g}_0$ annihilating an element in the quintic component of $H_2(\mathfrak{p}_+, \mathfrak{g})^1$;
- (2) given $\mathfrak{k} = \mathfrak{g}_- \oplus \mathfrak{k}_0$, to compute $H^{2,1}(\mathfrak{k}, \mathfrak{k})$, a Kuranishi family $(M_{\mathfrak{k}}, \xi_{\mathfrak{k}})$ and the quotient $M_{\mathfrak{k}}/G_0^{\mathfrak{k}}$;
- (3) given (\mathfrak{k}, ι) , to compute a regular, normal algebraic Cartan connection $\bar{\omega} : \mathfrak{k} \rightarrow \mathfrak{g}$ lifting ι ;
- (4) given $(\mathfrak{k}, \bar{\omega})$, to compute the harmonic curvature $\bar{\kappa}_m$ and symmetry dimension $d(m)$.

As we have remarked, whenever constructions are performed in families, we use linear algebra over a suitable ring. It will become clear later on that in case of a continuous family of conjugacy classes, say parameterised by a variable λ , one will first need to pass to the field $\mathbb{R}(\lambda)$ of rational functions, consider \mathfrak{k} as a Lie algebra over $\mathbb{R}(\lambda)$, then perform the construction of a Kuranishi family $M_{\mathfrak{k}}$ over $\mathbb{R}(\lambda)$, and further work in the ring of functions $\mathbb{R}(\lambda)[M_{\mathfrak{k}}]$ when computing algebraic Cartan connections. We do not devote any more space to the exploration of these issues.

Let us recall a standard trick facilitating the computation of Lie algebra cohomology: given a diagonalizable subalgebra $\mathfrak{a} \subset \mathfrak{k}$, we may restrict all considerations to the sub-complex of $C^\bullet(\mathfrak{k}, \mathfrak{k})$ annihilated by \mathfrak{a} : its \mathfrak{a} -invariant complement is homotopic to zero (this had been pointed out to me by Ian Anderson). Now, the Nijenhuis-Richardson bracket restricts to this sub-complex, and we thus have a sub-DGLA $\mathfrak{L}(\mathfrak{k})^\mathfrak{a}$ quasi-isomorphic to $\mathfrak{L}(\mathfrak{k})$. In particular, $H^1(\mathfrak{L}(\mathfrak{k})) = H^1(\mathfrak{L}(\mathfrak{k})^\mathfrak{a})$ and a Kuranishi family constructed for $\mathfrak{L}(\mathfrak{k})^\mathfrak{a}$ provides a Kuranishi family for $\mathfrak{L}(\mathfrak{k})$ itself. We also note that if $H_i^2(\mathfrak{L}(\mathfrak{k})) = 0$ for all $i \geq 2$, then $M_{\mathfrak{k}}$ is the entire $H^1(\mathfrak{L}(\mathfrak{k}))$ (there are no obstructions). In terms of Lie algebra cohomology, we have

$$H^{3,2}(\mathfrak{k}, \mathfrak{k}) = 0 \implies M_{\mathfrak{k}} = H^{2,1}(\mathfrak{k}, \mathfrak{k}).$$

Theorem 1. *The following algorithm yields a complete list, without repetitions, of all non-flat, at least 2-transitive homogeneous models of C_3 Monge geometries with vanishing scalar component of the harmonic curvature. First find the set of G_0 -conjugacy classes of graded subalgebras $\mathfrak{g}_- \subset \mathfrak{k} \subset \mathfrak{g}$ such that \mathfrak{k}_0 is at least two-dimensional, and annihilates a nonzero element of the quintic component of $H_2(\mathfrak{p}_+, \mathfrak{g})^1$. Then, for each class perform the following sequence:*

- (1) *Fix a representative \mathfrak{k} and compute a Kuranishi family $(M_{\mathfrak{k}}, \xi_{\mathfrak{k}})$.*
- (2) *Compute the family $(\mathfrak{k}_m)_{m \in M_{\mathfrak{k}}}$ of filtered deformations of \mathfrak{k} defined by $\xi_{\mathfrak{k}}$.*
- (3) *Find a family of regular, normal Cartan connections $\bar{\omega}_m : \mathfrak{k}_m \rightarrow \mathfrak{g}$, $m \in M_{\mathfrak{k}}$.*
- (4) *Compute the algebraic map*

$$\kappa_H : M_{\mathfrak{k}} \rightarrow H_2(\mathfrak{p}_+, \mathfrak{g})^1$$

such that $\kappa_H(m)$ is the harmonic curvature of ω_m .

- (5) *Compute the Zariski upper-semicontinuous map*

$$d : M_{\mathfrak{k}} \rightarrow \mathbb{Z}$$

such that $d(m)$ is the symmetry dimension of ω_m (Lemma 16).

- (6) *Let $M'_{\mathfrak{k}} \subset M_{\mathfrak{k}}$ be the algebraic subset cut out by the scalar component of κ_H , and $M''_{\mathfrak{k}} \subset M'_{\mathfrak{k}}$ its (possibly empty) intersection with the Zariski-open subset on which $d = \dim \mathfrak{k}$.*
- (7) *Compute the set-theoretic quotient $M''_{\mathfrak{k}}/G_0^{\mathfrak{k}}$ and append its points to the list of homogeneous models.*

Proof. Our goal is to describe the subset of $\pi_0(\mathbf{Germ})$ corresponding to non-flat, at least 2-transitive models with vanishing scalar component of the harmonic curvature. According to Lemmas 2 and 5 this is equivalent to describing the set of isomorphism classes of objects of the sub-groupoid of \mathbf{Def} consisting of objects (\mathfrak{k}, ι) satisfying the following properties:

- (1) the image of (\mathfrak{k}, ι) in \mathbf{Sub} is of the form $\mathfrak{k} = \mathfrak{g}_- \oplus \mathfrak{k}_0$ where \mathfrak{k}_0 is at least two-dimensional and annihilates a non-zero element of the quintic component of $H_2(\mathfrak{p}_+, \mathfrak{g})^1$,
- (2) the image of (\mathfrak{k}, ι) in $H_2(\mathfrak{p}_+, \mathfrak{g})^1 // G_0$ is a non-zero element of the quintic component of $H_2(\mathfrak{p}_+, \mathfrak{g})^1$,
- (3) the symmetry dimension of (\mathfrak{k}, ι) equals the dimension of \mathfrak{k} .

By Proposition 5 and Lemma 8, it is enough to restrict to a set of representatives of G_0 -conjugacy classes of graded subalgebras $\mathfrak{k} = \mathfrak{g}_- \oplus \mathfrak{k}_0$, and thus to a set of representatives of conjugacy classes of subalgebras $\mathfrak{k}_0 \subset \mathfrak{g}_0$.

More explicitly, denote by Σ a set consisting of precisely one representative for each conjugacy class of subalgebras $\mathfrak{k}_0 \subset \mathfrak{g}_0$ such that $\dim \mathfrak{k}_0 \geq 2$ and \mathfrak{k}_0 annihilates a non-zero element in the quintic component of $H_2(\mathfrak{p}_+, \mathfrak{g})^1$. Then the desired subset of $\pi_0(\mathbf{Germ})$ may be identified with the set of isomorphism classes of objects (\mathfrak{k}, ι) of **Def** such that:

- (1) $\iota(\mathfrak{k})_0 \in \Sigma$,
- (2) (as above)
- (3) the symmetry dimension of \mathfrak{k} equals $\dim \mathfrak{k}$.

Furthermore, since no two subalgebras in Σ are conjugate, any isomorphism between two such objects induces an isomorphism of their images in **Sub**, and thus is necessarily a morphism of $\mathbf{Def}_{\mathfrak{g}_- \oplus \mathfrak{k}_0}^*$ for some $\mathfrak{k}_0 \in \Sigma$. Now, for each $\mathfrak{k}_0 \in \Sigma$, letting $\mathfrak{k} = \mathfrak{g}_- \oplus \mathfrak{k}_0$, we have $\mathbf{Def}_{\mathfrak{k}}^* \approx M_{\mathfrak{k}} // G_0^{\mathfrak{k}}$ with a Kuranishi family $(M_{\mathfrak{k}}, \xi_{\mathfrak{k}})$ (Corollary 2 and Lemma 9). Hence the desired subset of $\pi_0(\mathbf{Germ})$ may be identified with the subset

$$\mathcal{M} \subset \coprod_{\mathfrak{k}_0 \in \Sigma} M_{\mathfrak{g}_- \oplus \mathfrak{k}_0} / G_0^{\mathfrak{g}_- \oplus \mathfrak{k}_0}$$

consisting of points $m \in M_{\mathfrak{k}}$ such that:

- (1) $\bar{\kappa}_{H,m}$ is a non-zero element of the quintic component of $H_2(\mathfrak{p}_+, \mathfrak{g})^1$,
- (2) the symmetry dimension of $(\mathfrak{k}_m, \bar{\omega}_m)$ computed by Lemma 16 equals $\dim \mathfrak{k}$,

where \mathfrak{k}_m is the deformed filtered Lie algebra structure on \mathfrak{k} defined by $\xi_{\mathfrak{k}}(m)$, and $\bar{\omega}_m : \mathfrak{k}_m \rightarrow \mathfrak{g}$ is a regular, normal algebraic Cartan connection with harmonic curvature $\bar{\kappa}_{H,m}$. A careful examination of the steps (1)-(7) of the algorithm shows that its outcome is precisely \mathcal{M} above. \square

One may view the above Theorem as reducing the classification problem of homogeneous models to the classification problem of subalgebras $\mathfrak{k}_0 \subset \mathfrak{g}_0$ with suitable properties. The success of this approach relies thus on being able to solve the latter task. In general, it is completely hopeless; here however, since our \mathfrak{g}_0 has semisimple rank 1, everything is as simple as the representation theory of $\mathfrak{sl}(2, \mathbb{R})$.

5. APPLICATION

5.1. Subalgebras. We are ready to begin implementing the above algorithm. Explicit calculations (cohomology, connections, symmetries) are performed using Ian Anderson's **DifferentialGeometry** package for MAPLE. Recall that $\mathfrak{g}_0 \simeq \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^2$. Let H, X, Y be a standard basis in the $\mathfrak{sl}(2, \mathbb{R})$ factor, with $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$. We may choose a basis E, E' in the abelian factor \mathbb{R}^2 so that E is the grading element in \mathfrak{g} , while E' annihilates the abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}_{-1}$ and acts as identity on \mathfrak{r} . Then, the quintic component in $H_2^1(\mathfrak{p}_+, \mathfrak{g})$ has weight 1 for E and 0 for E' . As a representation of $\mathfrak{sl}(2, \mathbb{R})$, it is identified with the space of quintic polynomials in $\mathbb{R}[z, w]$. We list G_0 -conjugacy classes of nonzero elements with at least two-dimensional stabiliser:

label	quintic	stabiliser
N	z^5	$\langle X, H - 5E, E' \rangle$
IV	$z^4 w$	$\langle H - 3E, E' \rangle$
F	$z^3 w^2$	$\langle H - E, E' \rangle$.

Lemma 18. *The following is a one-to-one enumeration of G_0 -conjugacy classes of graded subalgebras $\mathfrak{g}_- \subset \mathfrak{k} \subset \mathfrak{g}$ such that $\dim \mathfrak{k} \geq 10$ and \mathfrak{k} preserves a nonzero element in the quintic component of the harmonic curvature module:*

label	\mathfrak{k}_0
N_3	$\langle X, H - 5E, E' \rangle$
$N_{2a}^\lambda, \lambda \in \mathbb{RP}^1$	$\langle X, \lambda_0(H - 5E) + \lambda_1 E' \rangle$
N_{2b}	$\langle H - 5E, E' \rangle$
IV_2	$\langle H - 3E, E' \rangle$
F_2	$\langle H - E, E' \rangle$.

In each case, $\mathfrak{k} = \mathfrak{g}_- + \mathfrak{k}_0$.

Proof. The latter equality holds by Lemma 17, whence it remains to find G_0 -conjugacy classes of two-dimensional subalgebras \mathfrak{k}_0 in $\langle X, H - 5E, E' \rangle \subset \mathfrak{g}_0$. Choosing an element $Z \in \mathfrak{k}_0$ we have, up to conjugation, either $Z \in \langle H - 5E, E' \rangle$ (semisimple) or $Z = X$ (nilpotent). Thus, either $\mathfrak{k}_0 = \langle H - 5E, E' \rangle$ or it is spanned by X and a one-dimensional subspace in $\langle H - 5E, E' \rangle$. \square

We thus have four discrete classes: N_3, N_{2b}, IV_2, F_2 , as well as a one-parameter family $N_{2a}^\lambda, \lambda \in \mathbb{RP}^1$. When computing the Kuranishi families and algebraic Cartan connections for classes in the latter family, one needs to take care of the parameter λ . We will first discuss the discrete classes, as the computations are straightforward there.

5.2. Discrete cases. Recall that the Kuranishi family for the deformation problem associated with a given $\mathfrak{g}_- \subset \mathfrak{k} \subset \mathfrak{g}$ will be an algebraic subset $M_{\mathfrak{k}} \subset H^{2,1}(\mathfrak{k}, \mathfrak{k})^{\mathfrak{k} \cap \mathfrak{h}}$ together with a representing map $M_{\mathfrak{k}} \rightarrow C^{2,1}(\mathfrak{k}, \mathfrak{k})^{\mathfrak{k} \cap \mathfrak{h}}$ sending a point of $M_{\mathfrak{k}}$ to a suitable cochain ϕ satisfying the Maurer-Cartan condition. In the particular case of $H^{3,2}(\mathfrak{k}, \mathfrak{k}) = 0$, we have $M_{\mathfrak{k}} = H^{2,1}(\mathfrak{k}, \mathfrak{k})$; it will be the case for the four discrete classes. Indeed, we list the ‘Betti numbers’ $b_j^i = H_j^i(\mathfrak{k}, \mathfrak{k})$:

label	$b_j^2, j \geq 1$	$b_j^3, j \geq 2$
N_3	$1, 0, \dots$	$0, \dots$
N_{2b}	$1, 0, \dots$	$0, \dots$
IV_2	$2, 0, \dots$	$0, \dots$
F_2	$2, 0, \dots$	$0, \dots$

Thus, in all four cases the Kuranishi family is the entire affine space of dimension 1 or 2. Denoting the coordinates by t for N_3, N_{2b} and by t, s for IV_2, F_2 , we shall write down the representing map as a deforming cocycle $\phi(t)$ or $\phi(t, s)$, a polynomial with coefficients in $C^2(\mathfrak{k}, \mathfrak{k})$.

Let us introduce a convenient description of \mathfrak{g}_- . Recalling $\mathfrak{g}_0 \simeq \mathfrak{sl}(2, \mathbb{R}) \oplus \langle E, E' \rangle$, we have identifications

$$\begin{aligned} \mathfrak{g}_{-1} &\simeq \mathbb{R}^2[-1, 0] \oplus \mathbb{R}[-1, 1] \\ \mathfrak{g}_{-2} &\simeq \mathbb{R}^2[-2, 1] \\ \mathfrak{g}_{-3} &\simeq (S^2 \mathbb{R}^2)[-3, 1] \end{aligned}$$

where $\mathbb{R}^d[\lambda, \lambda']$ denotes the irreducible $U(\mathfrak{g}_0)$ -module of rank d on which E acts with weight λ , and E' with weight λ' . The structure equations are essentially determined by the requirement that they be homomorphisms of $U(\mathfrak{g}_0)$ -modules; normalization of the intertwiners may be absorbed into the above identifications. The $U(\mathfrak{g}_0)$ -module $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-3}^* \otimes \mathfrak{g}_{-3}$ contains a unique copy of $S^5 \mathbb{R}^2[1, 0]$, identified

with the quintic component of $H^2(\mathfrak{g}_-, \mathfrak{g})$. We let $v^N \in S^5\mathbb{R}^2[1, 0]$ be a non-zero vector annihilated by X , and set $v^{IV} = Yv^N$, $v^F = Y^2v^N$. These are the weight vectors in $S^5\mathbb{R}^2[1, 0]$ corresponding to the types N , IV and F . Using the $U(\mathfrak{g}_0)$ -equivariant projections, we may view them as maps $\Lambda^2\mathfrak{g}_- \rightarrow \mathfrak{g}_-$, and thus as elements of $C^2(\mathfrak{k}, \mathfrak{k})$ via the inclusion $\mathfrak{g}_- \subset \mathfrak{k}$.

The representing map for N_3 and N_{2b} may be immediately written as

$$\phi(t) = tv^N.$$

For types IV_2 and F_2 , we find a pair $v', v'' \in C^2(\mathfrak{k}, \mathfrak{k})^{\mathfrak{k} \cap \mathfrak{h}}$ representing a basis for $H^2(\mathfrak{k}, \mathfrak{k})$, and sharing the same $U(\mathfrak{h})$ -weight as v^{IV} , resp. v^F . For IV_2 , all Richardson-Nijenhuis brackets involving v', v'' vanish and the representing map is

$$\phi(t, s) = tv' + sv''.$$

For F_2 , there is an additional element $v''' \in C^2(\mathfrak{k}, \mathfrak{k})^{\mathfrak{k} \cap \mathfrak{h}}$ such that $dv''' + \frac{1}{2}[v', v'']$, and the representing map is

$$\phi(t, s) = tv' + sv'' + stv'''.$$

The other brackets are trivial. Note that in both IV_2 and F_2 , $\phi(t, s)$ is annihilated by entire \mathfrak{k}_0 .

We thus have cochains $\phi(t)$ or $\phi(t, s)$ defining one- or two-parameter families of filtered Lie algebras deforming \mathfrak{k} for each of the four discrete classes. The next step is to compute the regular, normal algebraic Cartan connection for every such deformation: this is done in terms of linear algebra over $\mathbb{R}(t)$ or $\mathbb{R}(t, s)$ and yields a map $\bar{\omega} : \mathfrak{k} \rightarrow \mathfrak{g}$ whose coefficients are polynomials in t or t, s . One may then find the harmonic curvature, which turns out to be contained in the quintic component for all values of the parameters t, s . Hence, $M'_\mathfrak{k} = M_\mathfrak{k}$ in all four classes.

Further, one computes the dimension of the symmetry algebra as in Lemma 16. This involves finding kernels of certain matrices over $\mathbb{R}(t)$ or $\mathbb{R}(t, s)$, and thus gives the symmetry dimension for the *generic* member of the deformation family in each class. The results follow:

label	N_3	N_{2b}	IV_2	F_2
generic d	11	11	10	10.

It turns out that the generic members of N_3 , IV_2 and F_2 are the actual symmetry algebras of the Monge geometries they define – hence, $M''_\mathfrak{k}$ is a dense Zariski open subset of $M_\mathfrak{k}$. On the other hand, by upper-semicontinuity, we have $d \geq 11$ in type N_{2b} so that the actual symmetry algebra is always larger than \mathfrak{k} in that type. Accordingly, $M''_\mathfrak{k} = \emptyset$ for N_{2b} and the entire family may be discarded.

We are left with three classes: N_3 , IV_2 and F_2 . The next step is to identify the action of $G_0^\mathfrak{k} \subset G_0$ on $M_\mathfrak{k}$ and describe the quotient $M''_\mathfrak{k}/G_0^\mathfrak{k}$. Recall that $G_0 \simeq \mathrm{SL}(2, \mathbb{R}) \times (\mathbb{R}^\times)^2$. Let $T \subset G_0$ denote the maximal torus with Lie algebra $\mathfrak{h} \subset \mathfrak{g}_0$. The subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is always T -invariant, so that $T \subset G_0^\mathfrak{k}$. Furthermore, all constructions related to the Kuranishi family are T -equivariant, so that $M_\mathfrak{k}$ and $M''_\mathfrak{k}$ are T -invariant subsets in $H^{2,1}(\mathfrak{k}, \mathfrak{k})$, and the representing map $\phi : M_\mathfrak{k} \rightarrow C^{2,1}(\mathfrak{k}, \mathfrak{k})^{\mathfrak{k} \cap \mathfrak{h}}$ is T -equivariant. It follows that the T -action on $M_\mathfrak{k}$ factors through the multiplicative action of \mathbb{R}^\times by means of the grading element (recall that v and v' have the same weight for $U(\mathfrak{h})$, hence for T). In coordinates, both t and s transform under \mathbb{R}^\times with weight 1. The identity component $G_0^{\mathfrak{k}, 0} \subset G_0^\mathfrak{k}$ and the associated quotient is then easy to describe:

label	N_3	IV_2	F_2
$G_0^{\mathfrak{k},0}$	$T \ltimes \exp\langle X \rangle$	T	T
$M_{\mathfrak{k}}/G_0^{\mathfrak{k},0}$	$\{0, *\}$	$\{0\} \cup \mathbb{RP}^1$	$\{0\} \cup \mathbb{RP}^1$

where 0 is the trivial deformation (hence flat), and $*$ is the unique non-flat model in N_3 .

This is already the minimal possible quotient for N_3 , so that there is precisely one homogeneous model of type N with 3-dimensional isotropy (up to equivalence). In the remaining types IV_2 and F_2 , we note that $G_0^{\mathfrak{k}}$ is contained in the normaliser $N_{G_0}T$ of the torus T in G_0 , while $G_0^{\mathfrak{k},0}$ is simply its centraliser, T . Now, $N_{G_0}T/T \simeq \mathbb{Z}_2$, the Weyl group of $\mathrm{SL}(2, \mathbb{R})$: its generator acts on \mathfrak{h} sending H to $-H$ and preserving both E and E' . Since \mathfrak{k}_0 is generated by E' and a *combination* of H and E , it follows that $G_0^{\mathfrak{k}}$ must actually preserve H , and thus coincides with the identity component $T = G_0^{\mathfrak{k}}$ of $N_{G_0}T$. An explicit computation of the algebraic Cartan connection shows that the harmonic curvature of the geometry defined by $\phi(s, t)$ is proportional to s , whence $s = 0$ is the unique point on $\mathbb{RP}^1 = (M_{\mathfrak{k}}/G_0^{\mathfrak{k},0}) \setminus \{0\}$ corresponding to a flat Monge geometry. Finally then, $M_{\mathfrak{k}}'/G_0\mathfrak{k}, 0$ embeds onto a dense open in $\mathbb{R} = \mathbb{RP}^1 \setminus \{s = 0\}$ parameterised by t/s .

5.3. Continuous family case. Let us now focus on the family N_{2a}^λ , $\lambda \in \mathbb{RP}^1$. Geometrically, one views the family of graded subalgebras $\mathfrak{k}_\lambda \subset \mathfrak{g}$ as a sub-bundle of the trivial vector bundle $\mathbb{RP}^1 \times \mathfrak{g}$ over \mathbb{RP}^1 . Correspondingly, the cochain complexes $C^\bullet(\mathfrak{k}_\lambda, \mathfrak{k}_\lambda)$ give a complex of vector bundles over \mathbb{RP}^1 . The differential and the Nijenhuis-Richardson bracket become homomorphisms of vector bundles, and in particular their fibre-wise images and kernels may have non-constant dimension. In the abstract deformation setting, we then have a bundle of DGLAs $\bigcup_\lambda \mathfrak{L}_\lambda \rightarrow \mathbb{RP}^1$, and we wish to construct Kuranishi families fibrewise. It is intuitively clear that, since all the data involved are of finite type, there should be a finite subset $\Lambda \subset \mathbb{RP}^1$ such that the calculations may be performed:

- separately for each $\lambda \in \Lambda$, and
- working over the field $\mathbb{R}(\lambda)$ for the complement $\mathbb{RP}^1 \setminus \Lambda$.

Indeed, the following provides the proper way to deal with N_{2a}^λ .

Lemma 19. *Identify \mathbb{RP}^1 with the projectivization of $\mathfrak{a} = \langle H - 5E, E' \rangle$, so that $\lambda = (\lambda_0, \lambda_1)$ corresponds to the line spanned by $\lambda_0(H - 5E) + \lambda_1 E'$. Set $\mathfrak{n} = \mathfrak{g} - \langle X \rangle$, so that $\mathfrak{k}_\lambda = \mathfrak{n} \oplus \lambda$ is the graded subalgebra of \mathfrak{g} corresponding to $\lambda \in \mathbb{RP}^1$. Let $\mathfrak{L}_\lambda = \mathfrak{L}(\mathfrak{k}_\lambda)$ and let $\mathfrak{L}_\lambda^{\mathfrak{a}} \subset \mathfrak{L}_\lambda$ denote the sub-DGLA annihilated by $\lambda \subset \mathfrak{k}_\lambda$. Recall that $\mathfrak{L}_\lambda^{\mathfrak{a}}$ and \mathfrak{L}_λ are equipped with an additional grading induced by that on \mathfrak{k}_λ .*

Let $\Sigma \subset \mathfrak{a}^ \setminus \{0\}$ be the set of nonzero weights of the $U(\mathfrak{a})$ -module $C^{\bullet,1}(\mathfrak{n}, \mathfrak{n}) \oplus C^{\bullet,1}(\mathfrak{n})$, and let $\Lambda \subset \mathbb{RP}^1$ be the union of the zero-loci of the linear forms in Σ . Then:*

- (1) $\bigcup_{\lambda \in \mathbb{RP}^1 \setminus \Lambda} \mathfrak{L}_\lambda^{\mathfrak{a}}$ forms a bundle of DGLA over $\mathbb{RP}^1 \setminus \Lambda$;
- (2) there is a DGLA \mathfrak{M} with additional grading, and an isomorphism

$$\bigcup_{\lambda \in \mathbb{RP}^1 \setminus \Lambda} \mathfrak{L}_\lambda^{\mathfrak{a}} \simeq \mathfrak{M} \times (\mathbb{RP}^1 \setminus \Lambda)$$

of bundles of DGLA over $\mathbb{RP}^1 \setminus \Lambda$, compatible with the additional gradings.

Proof. The result is more naturally cast in (real) algebro-geometric terms, where the family $(\mathfrak{L}_\lambda)_\lambda$ is identified with a coherent sheaf $\underline{\mathcal{K}}$ of graded Lie algebras, and $(\mathfrak{L}_\lambda)_\lambda$ with a coherent sheaf \mathcal{L} of DGLA on \mathbb{RP}^1 . Both are locally free as sheaves of $\mathcal{O}_{\mathbb{RP}^1}$ -modules, in particular

$$\underline{\mathcal{K}} \simeq \mathfrak{n} \otimes \mathcal{O}_{\mathbb{RP}^1} \oplus \mathcal{O}_{\mathbb{RP}^1}(-1)$$

with the second summand in graded degree zero. Now, $(\mathfrak{L}_\lambda^\mathfrak{a})_\lambda$ corresponds to the kernel in the exact sequence

$$0 \rightarrow \mathcal{L}^\mathfrak{a} \rightarrow \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{O}_{\mathbb{RP}^1}(1) = \mathcal{L}(1)$$

where the rightmost map is defined by the action of $\mathcal{O}_{\mathbb{RP}^1}(-1) \subset \underline{\mathcal{K}}$ on \mathcal{L} . Restricting over $\mathbb{RP}^1 \setminus \Lambda$, the inclusion $\mathcal{L}^\mathfrak{a} \rightarrow \mathcal{L}$ becomes split, so that in particular $\mathcal{L}^\mathfrak{a}$ is the sheaf of sections of a vector sub-bundle $\bigcup_{\lambda \notin \Lambda} \mathfrak{L}_\lambda^\mathfrak{a} \subset \bigcup_{\lambda \notin \Lambda} \mathfrak{L}_\lambda$. Let us choose a local trivialization

$$\underline{\mathcal{K}}|_{\mathbb{RP}^1 \setminus \Lambda} \simeq (\mathfrak{n} \oplus \mathbb{R}) \otimes \mathcal{O}_{\mathbb{RP}^1 \setminus \Lambda}$$

as graded locally free sheaves. There is an induced trivialization

$$\mathcal{L}|_{\mathbb{RP}^1 \setminus \Lambda} \simeq \tilde{\mathfrak{M}} \otimes \mathcal{O}_{\mathbb{RP}^1 \setminus \Lambda}, \quad \tilde{\mathfrak{M}} = C^{\bullet,1}(\mathfrak{n}, \mathfrak{n}) \oplus C^{\bullet-1,1}(\mathfrak{n}, \mathfrak{n}) \oplus C^{\bullet,1}(\mathfrak{n}) \oplus C^{\bullet-1,1}(\mathfrak{n})$$

as doubly graded locally free sheaves. We claim that there is a graded subspace $\mathfrak{M} \subset \tilde{\mathfrak{M}}$ equipped with a compatible structure of a DGLA, such that the above restricts to an isomorphism

$$\mathcal{L}_{\mathbb{RP}^1 \setminus \Lambda}^\mathfrak{a} \simeq \mathfrak{M} \otimes \mathcal{O}_{\mathbb{RP}^1 \setminus \Lambda}$$

of DGLA with additional grading. In fact \mathfrak{M} is precisely the subspace $\tilde{\mathfrak{M}}^\mathfrak{a} \subset \tilde{\mathfrak{M}}$ annihilated by \mathfrak{a} . Indeed, it is clear that $\tilde{\mathfrak{M}}^\mathfrak{a} \otimes \mathcal{O}_{\mathbb{RP}^1 \setminus \Lambda}$ is contained in the image of $\mathcal{L}_{\mathbb{RP}^1 \setminus \Lambda}^\mathfrak{a}$. Furthermore, the top row in the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\mathfrak{M}}^\mathfrak{a} \otimes \mathcal{O}_{\mathbb{RP}^1 \setminus \Lambda} & \longrightarrow & \tilde{\mathfrak{M}} \otimes \mathcal{O}_{\mathbb{RP}^1 \setminus \Lambda} & \longrightarrow & \tilde{\mathfrak{M}} \otimes \mathcal{O}_{\mathbb{RP}^1 \setminus \Lambda}(1) \\ & & \downarrow & & \parallel & & \parallel \\ 0 & \longrightarrow & \mathcal{L}^\mathfrak{a}|_{\mathbb{RP}^1 \setminus \Lambda} & \longrightarrow & \mathcal{L}|_{\mathbb{RP}^1 \setminus \Lambda} & \longrightarrow & \mathcal{L}(1)|_{\mathbb{RP}^1 \setminus \Lambda} \end{array}$$

is exact, since the only elements in $\tilde{\mathfrak{M}}$ annihilated by a subspace of \mathfrak{a} corresponding to a point of $\mathbb{RP}^1 \setminus \Lambda$ are those annihilated by entire \mathfrak{a} . Hence, the leftmost vertical arrow is an isomorphism. Finally, an inspection of the DGLA structure on $\mathcal{L}^\mathfrak{a}$ shows that the induced family of DGLA structures on $\mathfrak{M} = \tilde{\mathfrak{M}}^\mathfrak{a}$ is constant. \square

As a consequence, it is enough to compute the Kuranishi family for the finitely many deformations described by points of Λ , and for the generic point of \mathbb{RP}^1 . The set Λ includes $\infty = (0 : 1)$, so that its complement is an open subset of the affine line parameterised by $\lambda = \lambda_1/\lambda_0$. The computations for N_{2a}^λ , $\lambda \in \Lambda$ are performed as previously for the discrete classes. Then, for generic λ , we compute the Kuranishi family $M_{\mathfrak{M}}$ for the DGLA \mathfrak{M} as in Lemma 19. Considering $\bigcup_\lambda C^{2,1}(\mathfrak{L}_\lambda, \mathfrak{L}_\lambda)$ as a vector bundle over \mathbb{RP}^1 , the isomorphism of Lemma 19 gives the representing section

$$\Phi : M_{\mathfrak{M}} \times (\mathbb{RP}^1 \setminus \Lambda) \rightarrow \bigcup_{\lambda \notin \Lambda} C^{2,1}(\mathfrak{L}_\lambda, \mathfrak{L}_\lambda)$$

as an algebraic homomorphism of vector bundles over $\mathbb{RP}^1 \setminus \Lambda$. In fact, $M_{\mathfrak{M}}$ is an affine line; we parameterise it with a coordinate t . Since $\mathbb{RP}^1 \setminus \Lambda$ is contained in

the affine line $\mathbb{RP}^1 \setminus \{\infty\}$, the bundle $\bigcup_{\lambda \notin \Lambda} \mathfrak{k}_\lambda$ is trivial as a vector bundle, and isomorphic to $(\mathfrak{n} \oplus \mathbb{R}) \times (\mathbb{RP}^1 \setminus \Lambda)$. Using this trivialization, one checks that the representing map, written as

$$\Phi : \mathbb{R} \times (\mathbb{RP}^1 \setminus \Lambda) \rightarrow (\Lambda^2(\mathfrak{n} \oplus \mathbb{R})^* \otimes (\mathfrak{n} \oplus \mathbb{R})) \times (\mathbb{RP}^1 \setminus \Lambda)$$

is simply the constant lift of

$$\phi : \mathbb{R} \rightarrow \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n}, \quad \phi(t) = tv^N.$$

Still, the Lie algebra structure in the fibres of $\bigcup_{\lambda \notin \Lambda} \mathfrak{k}_\lambda$ depends algebraically on λ , and so does the deformed structure. One then computes a regular, normal algebraic Cartan connection as well as the symmetry dimension for the *generic* member in terms of linear algebra over $\mathbb{R}(t, \lambda)$. It turns out that the generic symmetry algebra is 11-dimensional. Since the dimension function $d : \mathbb{R} \times (\mathbb{RP}^1 \setminus \Lambda) \rightarrow \mathbb{Z}$ is Zariski upper-semicontinuous, it follows that $d \geq 11$ for all members of N_{2a}^λ , $\lambda \notin \Lambda$, whence the entire family may be discarded.

Turning to the finitely many models parameterised by Λ , it turns out that the only value of λ with nonempty M_λ'' is $\lambda = \infty$. We thus omit the calculations for the other elements of Λ , and focus on N_{2a}^∞ . The underlying graded subalgebra of \mathfrak{g} is $\mathfrak{k} = \mathfrak{g}_- \oplus \langle X, E' \rangle$. The interesting ‘Betti numbers’ read $b_1^2 = 3$, $b_2^2 = 1$, $b_i^2 = 0$ for $i \geq 3$. We introduce coordinates t_1, t_2, t_3 on $H_1^2(\mathfrak{k}, \mathfrak{k})$ and s on $H_2^2(\mathfrak{k}, \mathfrak{k})$. It turns out that this time $H^{3,2}(\mathfrak{k}, \mathfrak{k})$ is non-trivial and the Kuranishi family $M_\mathfrak{k} \subset H^{2,1}(\mathfrak{k}, \mathfrak{k})$ is the union of two irreducible components: the hyperplane $t_3 = 0$ and the line $t_1 = t_2 = s = 0$. Computing the algebraic Cartan connection and harmonic curvature for the generic points of the two components, one checks that deformations parameterised by the one-dimensional component give rise to flat Monge geometries. On the other hand, the value of the dimension function d at the generic point of the three-dimensional component is $d = 10$. We also have $M'_\mathfrak{k} = M_\mathfrak{k}$ so that $M_\mathfrak{k}''$ is a dense open subset of the three-dimensional irreducible component parameterised by t_1, t_2, s . The curvature of our representative connection is harmonic and proportional to t_1 , so that in particular $M_\mathfrak{k}''$ is contained in the subset $t_1 \neq 0$.

We now need to pass to the quotient by the action of $G_0^\mathfrak{k}$. As before, one easily describes $G_0^{\mathfrak{k},0} = T \times \exp\langle X \rangle$. The Kuranishi family and representing map are T -equivariant, and the weights of t_1, t_2, s under the action of T are given by:

$$\begin{aligned} e^{aH}(t_1, t_2, s) &= (e^{5a}t_1, e^a t_2, e^{2a}s) \\ e^{aE}(t_1, t_2, s) &= (e^a t_1, e^a t_2, e^{2a}s) \\ e^{aE'}(t_1, t_2, s) &= (t_1, t_2, s). \end{aligned}$$

One checks by explicit calculation that the remaining unipotent factor $\exp\langle X \rangle$ acts trivially on the three-dimensional component of $M_\mathfrak{k}$, i.e. that the action of X on the cochain parameterised by t_1, t_2, s corresponds to the gauge action of a suitable element of $C^{1,1}(\mathfrak{k}, \mathfrak{k})$. Now, $M_\mathfrak{k}''$ is contained in the complement of $t_1 = 0$, whose image in $\mathbb{R}^3/G_0^{\mathfrak{k},0}$ may be described by normalising $t_1 = 1$ and taking the quotient of \mathbb{R}^2 , parameterised by t_2, s , by the rank one sub-torus $\mathbb{R}^\times \subset T$ generated by $H - 5E$. This latter torus acts on t_2 with weight 4, and on s with weight 8. It follows that $M_\mathfrak{k}''/G_0^{\mathfrak{k},0}$ is an open subset of $\mathbb{R}^2/\mathbb{R}^\times \simeq \{*\} \cup (-\infty, \infty) \times \{\pm 1\} \cup \{\pm\infty\}$, where $*$ is the singleton orbit of $t_2 = s = 0$, the endpoints $\pm\infty$ parameterise half-lines $t_2 = 0, s > 0$ and $t_2 = 0, s < 0$, while $(-\infty, \infty) \times \{\pm 1\}$ is the set of half-parabolas $s = \alpha t_2^2, t_2 = \epsilon$ with $\alpha \in \mathbb{R}, \epsilon = \pm 1$. The point $*$, corresponding to $t_1 = 1$,

$t_2 = s = 0$, gives rise to a geometry with 11-dimensional symmetry of type N_3 , thus $M_{\mathfrak{k}}''/G_0^{\mathfrak{k},0} \subset (-\infty, \infty) \times \{\pm 1\} \cup \{\pm \infty\}$. It is convenient to visualise the latter space as two parallel copies of the real line, sharing the endpoints $\pm \infty$. Finally, it is easy to observe that the full subgroup of $\mathrm{SL}(2, \mathbb{R})$ preserving both H and $\langle X \rangle$ is connected, and so is thus $G_0^{\mathfrak{k}}$. Hence, $M_{\mathfrak{k}}''/G_0^{\mathfrak{k}}$ is a dense open subset of $(-\infty, \infty) \times \{\pm 1\} \cup \{\pm \infty\}$, parameterised by s/t_2^2 and the sign of t_2 . An explicit computation of the symmetry algebra dimension for the endpoints $\pm \infty$ shows that they do belong to $M_{\mathfrak{k}}''/G_0^{\mathfrak{k}}$.

5.4. Final result. We thus arrive at the final list of homogeneous models. We express the structure equations in terms of the exterior derivatives of the dual basis of left-invariant forms on a Lie group K integrating \mathfrak{k} . The vertical distribution on $K \rightarrow K/L$ is the annihilator of $\theta_1, \dots, \theta_8$, while the preimage in TK of the Monge distribution is the annihilator of $\theta_1, \dots, \theta_5$.

- Type N_3 .

$$\begin{aligned}
d\theta^1 &= -2\theta^1 \wedge \theta^9 - 2\theta^1 \wedge \theta^{11} - 2\theta^4 \wedge \theta^6 \\
d\theta^2 &= -\theta^1 \wedge \theta^{10} - 2\theta^2 \wedge \theta^{11} - \theta^4 \wedge \theta^7 - \theta^5 \wedge \theta^6 \\
d\theta^3 &= -\theta^1 \wedge \theta^6 - 2\theta^2 \wedge \theta^{10} + 2\theta^3 \wedge \theta^9 - 2\theta^3 \wedge \theta^{11} - 2\theta^5 \wedge \theta^7 \\
d\theta^4 &= -6\theta^4 \wedge \theta^9 - 2\theta^4 \wedge \theta^{11} + \theta^6 \wedge \theta^8 \\
d\theta^5 &= -\theta^4 \wedge \theta^{10} - 4\theta^5 \wedge \theta^9 - 2\theta^5 \wedge \theta^{11} + \theta^7 \wedge \theta^8 \\
d\theta^6 &= 4\theta^6 \wedge \theta^9 \\
d\theta^7 &= -\theta^6 \wedge \theta^{10} + 6\theta^7 \wedge \theta^9 \\
d\theta^8 &= -10\theta^8 \wedge \theta^9 - 2\theta^8 \wedge \theta^{11} \\
d\theta^9 &= 0 \\
d\theta^{10} &= -2\theta^9 \wedge \theta^{10} \\
d\theta^{11} &= 0
\end{aligned}$$

- Type N_{2a}^∞ , $\epsilon = \pm 1$, $\alpha \in \mathbb{R}$ generic (interior).

$$\begin{aligned}
d\theta^1 &= \epsilon\theta^1 \wedge \theta^6 - 2\theta^1 \wedge \theta^{10} - 2\theta^4 \wedge \theta^6 \\
d\theta^2 &= -\theta^1 \wedge \theta^9 - 2\theta^2 \wedge \theta^{10} - \theta^4 \wedge \theta^7 - \theta^5 \wedge \theta^6 \\
d\theta^3 &= \theta^1 \wedge \theta^6 - 2\theta^2 \wedge \theta^9 - \epsilon\theta^3 \wedge \theta^6 - 2\theta^3 \wedge \theta^{10} - 2\theta^5 \wedge \theta^7 \\
d\theta^4 &= \alpha\theta^1 \wedge \theta^6 - 2\theta^4 \wedge \theta^{10} + \theta^6 \wedge \theta^8 \\
d\theta^5 &= \alpha\theta^2 \wedge \theta^6 - \theta^4 \wedge \theta^9 - \epsilon\theta^5 \wedge \theta^6 - 2\theta^5 \wedge \theta^{10} + \theta^7 \wedge \theta^8 \\
d\theta^6 &= 0 \\
d\theta^7 &= -\theta^6 \wedge \theta^9 \\
d\theta^8 &= 2\alpha\theta^4 \wedge \theta^6 + \epsilon\theta^6 \wedge \theta^8 - 2\theta^8 \wedge \theta^{10} \\
d\theta^9 &= \alpha\theta^6 \wedge \theta^7 + \epsilon\theta^6 \wedge \theta^9 \\
d\theta^{10} &= 0
\end{aligned}$$

- Type N_{2a}^∞ , $\sigma = \pm 1$ (boundary).

$$\begin{aligned}
d\theta^1 &= -2\theta^1 \wedge \theta^{10} - 2\theta^4 \wedge \theta^6 \\
d\theta^2 &= -\theta^1 \wedge \theta^9 - 2\theta^2 \wedge \theta^{10} - \theta^4 \wedge \theta^7 - \theta^5 \wedge \theta^6 \\
d\theta^3 &= \theta^1 \wedge \theta^6 - 2\theta^2 \wedge \theta^9 - 2\theta^3 \wedge \theta^{10} - 2\theta^5 \wedge \theta^7 \\
d\theta^4 &= \sigma\theta^1 \wedge \theta^6 - 2\theta^4 \wedge \theta^{10} + \theta^6 \wedge \theta^8 \\
d\theta^5 &= \sigma\theta^2 \wedge \theta^6 - \theta^4 \wedge \theta^9 - 2\theta^5 \wedge \theta^{10} + \theta^7 \wedge \theta^8 \\
d\theta^6 &= 0 \\
d\theta^7 &= -\theta^6 \wedge \theta^9 \\
d\theta^8 &= 2\sigma\theta^4 \wedge \theta^6 - 2\theta^8 \wedge \theta^{10} \\
d\theta^9 &= \sigma\theta^6 \wedge \theta^7 \\
d\theta^{10} &= 0
\end{aligned}$$

- Type IV_2 , $\alpha \in \mathbb{R}$ generic.

$$\begin{aligned}
d\theta^1 &= -2\theta^1 \wedge \theta^9 - 2\theta^1 \wedge \theta^{10} - 2\theta^4 \wedge \theta^6 \\
d\theta^2 &= (2 + 2\alpha)\theta^1 \wedge \theta^6 - 2\theta^2 \wedge \theta^{10} - \theta^4 \wedge \theta^7 - \theta^5 \wedge \theta^6 \\
d\theta^3 &= 2\alpha\theta^2 \wedge \theta^6 + 2\theta^3 \wedge \theta^9 - 2\theta^3 \wedge \theta^{10} - 2\theta^5 \wedge \theta^7 \\
d\theta^4 &= -4\theta^4 \wedge \theta^9 - 2\theta^4 \wedge \theta^{10} + \theta^6 \wedge \theta^8 \\
d\theta^5 &= \alpha\theta^4 \wedge \theta^6 - 2\theta^5 \wedge \theta^9 - 2\theta^5 \wedge \theta^{10} + \theta^7 \wedge \theta^8 \\
d\theta^6 &= 2\theta^6 \wedge \theta^9 \\
d\theta^7 &= 4\theta^7 \wedge \theta^9 \\
d\theta^8 &= -6\theta^8 \wedge \theta^9 - 2\theta^8 \wedge \theta^{10} \\
d\theta^9 &= 0 \\
d\theta^{10} &= 0
\end{aligned}$$

- Type F_2 , $\alpha \in \mathbb{R}$ generic.

$$\begin{aligned}
d\theta^1 &= (1 + 2\alpha)\theta^1 \wedge \theta^6 - 2\theta^1 \wedge \theta^9 - 2\theta^1 \wedge \theta^{10} - 2\theta^4 \wedge \theta^6 \\
d\theta^2 &= \alpha\theta^1 \wedge \theta^7 - 2\theta^2 \wedge \theta^{10} - \theta^4 \wedge \theta^7 - \theta^5 \wedge \theta^6 \\
d\theta^3 &= 2\theta^3 \wedge \theta^9 - 2\theta^3 \wedge \theta^{10} - 2\theta^5 \wedge \theta^7 \\
d\theta^4 &= \alpha\theta^1 \wedge \theta^6 - 2\theta^4 \wedge \theta^9 - 2\theta^4 \wedge \theta^{10} + \theta^6 \wedge \theta^8 \\
d\theta^5 &= 2\alpha\theta^5 \wedge \theta^6 - 2\theta^5 \wedge \theta^{10} + \theta^7 \wedge \theta^8 \\
d\theta^6 &= 0 \\
d\theta^7 &= 2\alpha\theta^6 \wedge \theta^7 + 2\theta^7 \wedge \theta^9 \\
d\theta^8 &= -4\alpha\theta^6 \wedge \theta^8 - 2\theta^8 \wedge \theta^9 - 2\theta^8 \wedge \theta^{10} \\
d\theta^9 &= 0 \\
d\theta^{10} &= 0
\end{aligned}$$

6. COMPARISON TO THE WORK OF ANDERSON AND NUROWSKI

6.1. We may now compare our results to those of Anderson and Nurowski in [2]. As we have already remarked, the geometries they consider are restricted to type N , but the classification is carried through all the way down to simply-transitive models. Hence, their list intersects ours in the 11-dimensional model of type N_3 , as well as in the one-parameter family of 10-dimensional models N_{2a}^∞ . It is indeed

possible to find an explicit equivalence between the realisations written down in the two papers.

6.2. Naturally, the possibility of such comparison prompts the question about the advantages and disadvantages of each of the two methods, i.e. deformation theory in our case and Cartan reduction in [2]. We shall only contrast the two techniques in a couple of aspects. First, we note that Cartan reduction, if properly performed, gives a much more efficient algorithm in the sense that it necessarily produces a complete list of inequivalent models. Furthermore, since assumptions on the possible normalisations of structure functions are built into the very core of the procedure, it is straightforward to restrict the classification to a given curvature type (e.g. type N , as Anderson and Nurowski have done), with a guarantee that no examples of other types are produced.

Our method gives no such control over the curvature; while we do use assumptions on the curvature type to produce a list of classes of isotropy subalgebras $\mathfrak{k}_0 \subset \mathfrak{g}_0$, it may well happen that a particular component of the Kuranishi space deforms the flat algebra into a different curvature type than intended (this had been observed for some of the 1-transitive models). We are thus in general forced to filter our raw list of models. Likewise, when parameterising deformations belonging to a given component of a Kuranishi space for some \mathfrak{k} , we ought to carefully identify (1) intersections with other components, and (2) points that should be glued to a Kuranishi space for a larger $\mathfrak{k}' \supset \mathfrak{k}$. For instance, comparing our family N_{2a}^∞ to the one found by Anderson and Nurowski, we see that certain special values of the parameter α should be excluded as defining models with a larger symmetry algebra (recall that we compute the dimension of the symmetry algebra only at the generic point of each irreducible component).

By design, these problems do not arise in the approach based on Cartan reduction. However, a careful reading of [2] shows that achieving such efficiency requires a great deal of experience and technical skill. This is in contrast to the deformation-theoretic approach, which comes with the advantage of using standard cohomological tools susceptible to far-reaching automation, and may be applied without a deep understanding of the invariants of a geometry in question. In future applications one might perhaps use the deformation-theoretic method algorithmically to produce an early survey of the terrain, only to be elaborated by means of Cartan reduction.

6.3. Let us finally remark that our method may be in principle applied also to produce 1-transitive homogeneous models of the C_3 Monge geometry. The computations in that case become too large to handle without first developing more subtle algorithms. To give the reader a taste of the situation, let us mention that the graded subalgebras $\mathfrak{k} \subset \mathfrak{g}$ in this case include the *nilpotent* algebra $\mathfrak{g}_- \oplus \langle X \rangle$, admitting a 10-dimensional $H^{2,1}(\mathfrak{k}, \mathfrak{k})$ and a Kuranishi family with 27 irreducible components. Interestingly enough, all those components, and indeed all the irreducible components of all Kuranishi spaces we have encountered in this project, turn out to be rational.

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